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RADIATIVE EQUILIBRIUM IN AN ATMOSPHERE WITH CONSTANT EINSTEIN ABSORPTION COEFFICIENT

by Rupert Wildt and Sandra Schwartz

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ABSTRACT

In the classical gray atmosphere, the phenomenological coefficient of absorption that enters into Kirchhoff's law is assumed to be independent of the frequency (κ_ν = const = $\bar{\kappa}$). Strictly considered, this coefficient is the product obtained by multiplying the Einstein coefficient of true absorption by a frequency-dependent correction for induced emission (Rosseland factor). The alternative here examined is to set the Einstein coefficient constant, incorporate the effects of induced emission into the transfer formalism, and determine the march of the local thermodynamic equilibrium (LTE) source function by numerical iteration. This second kind of gray atmosphere differs markedly from the classical one in respect to many physical characteristics and represents, in conventional terminology, the simplest non-gray problem that is physically realistic.

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INTRODUCTION

The theory of radiative transfer through the gray atmosphere antedates the concept of induced emission and the ensuing distinction (Reference 1) between two kinds of phenomenological coefficients of absorption [dimension (cm^{-1})]. The first of these is properly called the Kirchhoff coefficient, because it appears in Kirchhoff's law, and the assumption that it be gray (κ_ν = const = $\bar{\kappa}$) is fundamental to the classical theory of κ. Schwarzschild (Reference 2) and its elaboration by Milne, Hopf, and others. The second one (Reference 3), which measures the attenuation by true absorption but excludes the effect of induced emission (negative absorption), will here be denoted by α_ν and called the Einstein coefficient. Owing to the exclusion noted, the inequality $\kappa_\nu < \alpha_\nu$ holds generally. The equation

$$\kappa_\nu = \alpha_\nu \left[1 - \exp\left(-\frac{k\nu}{kT}\right)\right], \tag{1}$$

while true for thermodynamic equilibrium (Reference 1), is unnecessarily restrictive (Reference 4). A relation of unrestricted validity,

$$\kappa_\nu = \alpha_\nu - \frac{c^2 \epsilon_\nu}{2h\nu^3}, \tag{2}$$

where $\epsilon_\nu d\nu$ (erg/cm^3-sec-ster) is the phenomenological coefficient of spontaneous (isotropic) emission, follows from the most general, time-dependent form of the equation of energy transfer in Cartesian coordinates,

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mu_1 \frac{\partial I_\nu}{\partial x_1} + \mu_2 \frac{\partial I_\nu}{\partial x_2} + \mu_3 \frac{\partial I_\nu}{\partial x_3} = -\alpha_\nu I_\nu + \epsilon_\nu + \frac{c^2 I_\nu}{2h\nu^3} \epsilon_\nu , \tag{3}$$

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in which the first term on the right-hand side represents the rate of true absorption, the second term the rate of spontaneous emission, and the third term the rate of induced emission, which must occur precisely in the direction of the incident pencil and at a rate proportional to its intensity, according to Einstein's analysis of the momentum transfer. By rearranging the terms on the right-hand side of Equation 3, the conventional form of the energy transfer equation is recovered,

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mu_1 \frac{\partial I_\nu}{\partial x_1} + \mu_2 \frac{\partial I_\nu}{\partial x_2} + \mu_3 \frac{\partial I_\nu}{\partial x_3} = -\left(\alpha_\nu - \frac{c^2 \epsilon_\nu}{2h\nu^3}\right) I_\nu + \epsilon_\nu \ , \tag{4}$$

and the relation (2) is thus seen to apply even to non-isotropic and transient radiation fields. A more transparent form can be obtained by introducing the familiar definition of the source function,

$$\frac{\epsilon_\nu}{\kappa_\nu} \equiv I_\nu^* \ , \tag{5}$$

and writing

$$\kappa_\nu = \alpha_\nu \left(1 + \frac{c^2 I_\nu^*}{2h\nu^3}\right)^{-1} . \tag{6}$$

Instead of using I_ν^*, the spectrum of the source function can be characterized by a reciprocal monochromatic temperature,

$$\frac{1}{T_\nu^*} = \frac{k}{h\nu} \ln \left(1 + \frac{2h\nu^3}{c^2 I_\nu^*}\right) , \tag{7}$$

which will generally be frequency-dependent. An equivalent version of Equation 6, then, is

$$\kappa_\nu = \alpha_\nu \left[1 - \exp\left(-\frac{h\nu}{kT_\nu^*}\right)\right] . \tag{8}$$

The great interest attaching to Equations 6 and 8 derives from the fact that they rest on the strictly formal nature of the definitions 5 and 7 which gives them validity for any spectral distribution of the source function.

In the special case of LTE, Kirchhoff's law holds, by hypothesis, so the right-hand side of Equation 5 becomes the Planck function:

$$I_\nu^* = \frac{2h\nu^3}{c^2}\left[\exp \frac{h\nu}{k}\left(\frac{\sigma}{\pi I^*}\right)^{\frac{1}{4}} - 1\right]^{-1} , \qquad I^* = \int_0^\infty I_\nu^* \, d\nu \ , \tag{9}$$

where σ is the Stefan-Boltzmann constant. In this case, both sides of Equation 7 are independent of the frequency,

$$\frac{1}{T_\nu^*} = \left(\frac{\sigma}{\pi I^*}\right)^{\frac{1}{4}} = \frac{1}{T}, \tag{10}$$

and Equation 8 reduces to Rosseland's relation, (Equation 1). If, in addition to the postulate of LTE, it is assumed that the Kirchhoff absorption coefficient is independent of the frequency (classical gray atmosphere with $\kappa_\nu = \text{const} = \bar{\kappa}$), then it follows necessarily from the relation

$$\bar{\kappa} = \alpha_\nu \left\{1 - \exp\left[-\frac{h\nu}{k}\left(\frac{\sigma}{\pi I^*}\right)^{\frac{1}{4}}\right]\right\} \tag{11}$$

that the concealed mechanism of induced emission implies

$$\lim_{\nu \to 0} \alpha_\nu = \infty \tag{12}$$

and that the rate at which the Einstein coefficient tends to this limit varies from level to level in the atmosphere, depending on the local value of I^*. As in the classical model of a semi-infinite atmosphere, I^* itself tends to infinity with increasing depth; similarly, at any finite frequency, the Einstein coefficient, α_ν, tends to infinity. The presumed simplicity of the classical concept of gray matter turns out to be specious when due attention is paid to the phenomenon of induced emission. Conceding that the classical gray atmosphere is a rather artificial model detracts in no way from the value of the study lavished on it. It does suggest, however, examination of a second kind of gray atmosphere, distinguished by postulating the Einstein coefficient to be independent of frequency; in other words, let it be assumed that

$$\alpha_\nu = \text{constant} = \bar{\alpha} \tag{13}$$

and

$$\kappa_\nu = \bar{\alpha}\left(1 + \frac{c^2 I_\nu^*}{2h\nu^3}\right)^{-1}, \tag{14}$$

or

$$\kappa_\nu = \bar{\alpha}\left[1 - \exp\left(-\frac{h\nu}{kT_\nu^*}\right)\right]. \tag{15}$$

Radiative transfer through such an "Einstein-gray" atmosphere poses, perhaps, the simplest nongray problem (in conventional terminology) that is physically realistic.

The well-known formulary of non-gray radiative transfer in a plane parallel atmosphere is readily adapted for use in the Einstein-gray case by introducing as independent variable the *neutral optical depth* defined by

$$\tau = \int_x^\infty \bar{\alpha}(\xi)\, d\xi \; . \tag{16}$$

Moreover, by Equation 14,

$$d\tau_\nu = \left(1 + \frac{c^2 I_\nu^*}{2h\nu^3}\right)^{-1} d\tau \; , \tag{17}$$

and the optical depth that measures the monochromatic attenuation of a pencil escaping from level τ below the surface is

$$\tau_\nu(\tau) = \int_0^\tau \left[1 + \frac{c^2 I_\nu^*(t)}{2h\nu^3}\right]^{-1} dt \; . \tag{18}$$

Then the condition of strict radiative equilibrium at the *neutral level* reads as follows:

$$\int_0^\infty \left[1 + \frac{c^2 I_\nu^*(\tau)}{2h\nu^3}\right]^{-1} \left\{ I_\nu^*(\tau) - \frac{1}{2}\int_\tau^\infty \left[1 + \frac{c^2 I_\nu^*(t)}{2h\nu^3}\right]^{-1} I_\nu^*(t)\, E_1\left[t_\nu(t) - \tau_\nu(\tau)\right] dt \right.$$
$$\left. - \frac{1}{2}\int_0^\tau \left[1 + \frac{c^2 I_\nu^*(t)}{2h\nu^3}\right]^{-1} I_\nu^*(t)\, E_1\left[\tau_\nu(\tau) - t_\nu(t)\right] dt \right\} d\nu = 0 = -\frac{1}{4}\frac{dF(\tau)}{d\tau} \; , \tag{19}$$

where $F(\tau)$ is the net radiant flux at the level τ. This is the analogue of Milne's integral equation for the classical gray atmosphere. The use of the symbols I_ν^* or T_ν^* in Equations 14 through 19 stresses the fact that they hold in the absence of LTE.

THE EINSTEIN-GRAY ATMOSPHERE IN LTE

When the Planck function (Equation 9) for I_ν^* is substituted in Equation 19, the first term inside the braces (multiplied by the preceding Rosseland factor) can be made integrable over the entire spectrum, so that the integral is directly proportional to the *total* source function $I^*(\tau)$. This method provides no such simplification for the second and third terms, because the monochromatic optical depths appearing as arguments of the exponential integrals, E_1, preclude separation of the integrations with respect to ν and t. Since this LTE problem is non-gray in the conventional (Kirchhoff) sense, there is small hope of finding an exact analytical solution. Therefore, it has been thought worthwhile to try an iteration scheme in order to obtain a numerical solution.

After reverting to the familiar symbol $B_\nu(\tau)$ for the LTE source function, and writing

$$B_\nu(\tau) = \frac{2h\nu^3}{c^2}\left\{\exp \frac{h\nu}{k}\left[\frac{\sigma}{\pi B(\tau)}\right]^{\frac{1}{4}} - 1\right\}^{-1},$$

$$B(\tau) = \int_0^\infty B_\nu(\tau)\, d\nu , \qquad T(\tau) = \left[\pi B(\tau)/\sigma\right]^{\frac{1}{4}} , \tag{20}$$

the problem to be solved is to find the total source function $B(\tau)$ satisfying the integral equation expressing strict radiative equilibrium, or constancy of the net flux πF of the total radiant energy, in a gray atmosphere of the second kind, namely,

$$-\frac{1}{4}\frac{dF}{d\tau} = \int_0^\infty \left(1 - \exp\left\{-\frac{h\nu}{k}\left[\frac{\sigma}{\pi B(\tau)}\right]^{\frac{1}{4}}\right\}\right)\left[B_\nu(\tau) - J_\nu(\tau)\right] d\nu = 0, \qquad 0 \leqq \tau < \infty , \tag{21}$$

in which $B_\nu(\tau)$ is given by Equation 20, and $J_\nu(\tau)$ is the mean monochromatic intensity. The latter is related to $B_\nu(\tau)$ by Hopf's Λ-operator. Since the Rosseland factor appears in the monochromatic optical depths entering the Λ-operator, its explicit form in this case is

$$\Lambda_\tau\left[B_\nu(t)\right] = \frac{1}{2}\int_0^\infty B_\nu(t)\left(1 - \exp\left\{-\frac{h\nu}{k}\left[\frac{\sigma}{\pi B(t)}\right]^{\frac{1}{4}}\right\}\right) E_1\left(\left|\tau_\nu(\tau) - t_\nu(t)\right|\right) dt = J_\nu(\tau) , \tag{22}$$

with

$$\tau_\nu(\tau) = \int_0^\tau \left(1 - \exp\left\{-\frac{h\nu}{k}\left[\frac{\sigma}{\pi B(t)}\right]^{\frac{1}{4}}\right\}\right) dt \tag{23}$$

and

$$t_\nu(t) = \int_0^t \left(1 - \exp\left\{-\frac{h\nu}{k}\left[\frac{\sigma}{\pi B(s)}\right]^{\frac{1}{4}}\right\}\right) ds . \tag{24}$$

The condition of flux constancy,

$$F(\tau) = \int_0^\infty F_\nu(\tau)\, d\nu = \text{constant} , \qquad 0 \leqq \tau < \infty , \tag{25}$$

$$F_\nu(\tau) = 2\int_\tau^\infty B_\nu(t)\left(1 - \exp\left\{-\frac{h\nu}{k}\left[\frac{\sigma}{\pi B(t)}\right]^{\frac{1}{4}}\right\}\right) E_2\left[t_\nu(t) - \tau_\nu(\tau)\right] dt$$

$$- 2\int_0^\tau B_\nu(t)\left(1 - \exp\left\{-\frac{h\nu}{k}\left[\frac{\sigma}{\pi B(t)}\right]^{\frac{1}{4}}\right\}\right) E_2\left[\tau_\nu(\tau) - t_\nu(t)\right] dt \ , \qquad (26)$$

is equivalent to the integral Equation 21.

This step completes the statement of the problem. Its numerical solution will be carried out by adapting to the problem at hand an iteration scheme due to Lecar (Reference 5).

THE LINEAR APPROXIMATION TO THE SOURCE FUNCTION: $B^{(0)}(\tau)$

The first step is to find a linear approximation to the source function that will yield the asymptotic value at large optical depths of the prescribed total net flux.

As Lecar has proved, by expanding $B_\nu(t_\nu)$ in a Taylor series about the point $t_\nu = \tau_\nu$,

$$\lim_{\tau\to\infty} F_\nu(\tau) = \frac{4}{3}\lim_{\tau\to\infty}\frac{\partial B_\nu(\tau_\nu)}{\partial\tau_\nu} = \frac{4}{3}\lim_{\tau\to\infty}\frac{\partial B_\nu}{\partial T}\frac{dT}{d\tau}\frac{\partial\tau}{\partial\tau_\nu} \ , \qquad (27)$$

provided that

$$\lim_{\tau_\nu\to\infty}\frac{\partial^n B_\nu(\tau_\nu)}{\partial\tau_\nu^{\,n}} = 0 \ , \qquad n \geqq 3 \ . \qquad (28)$$

According to Equation 24

$$\frac{\partial\tau}{\partial\tau_\nu} = \left[1 - \exp\left(-\frac{h\nu}{kT(\tau)}\right)\right]^{-1} . \qquad (29)$$

Therefore

$$\frac{4}{3}\lim_{\tau\to\infty}\frac{dT}{d\tau}\int_0^\infty \frac{\partial B_\nu}{\partial T}\left[1 - \exp\left(-\frac{h\nu}{kT}\right)\right]^{-1} d\nu = F \ , \qquad (30)$$

where F is the prescribed value of the total radiative flux.

With k_r defined by

$$\frac{1}{k_r}\frac{dB}{dT} = \int_0^\infty \frac{\partial B_\nu}{\partial T}\left[1 - \exp\left(-\frac{h\nu}{kT}\right)\right]^{-1} d\nu \ , \tag{31}$$

Equation 30 becomes

$$\frac{4}{3}\lim_{\tau\to\infty}\frac{1}{k_r}\frac{dB}{dT}\frac{dT}{d\tau} = \frac{4}{3}\lim_{\tau\to\infty}\frac{1}{k_r}\frac{dB}{d\tau} = F \ . \tag{32}$$

Substituting in Equation 31 the following expressions,

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3} \ , \tag{33}$$

$$\frac{dB}{dT} = \frac{8\pi^4 k^4}{15c^2 h^3} T^3 \ , \tag{34}$$

and

$$\frac{\partial B_\nu}{\partial T} = \frac{2h^2 \nu^4}{c^2 kT^2}\exp\left(\frac{h\nu}{kT}\right)\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-2} \ , \tag{35}$$

shows that

$$\frac{1}{k_r} = \frac{15}{4\pi^4}\frac{h^5}{k^5 T^5}\int_0^\infty \nu^4 \exp\left(\frac{2h\nu}{kT}\right)\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-3} d\nu \ . \tag{36}$$

In terms of the variable x, defined by

$$x = \frac{h\nu}{kT} \tag{37}$$

and

$$dx = \frac{h}{kT}\, d\nu \ , \tag{38}$$

$$\frac{1}{k_r} = \frac{15}{4\pi^4}\int_0^\infty \frac{x^4 e^{2x}}{\left(e^x - 1\right)^3}\, dx \ . \tag{39}$$

Integration by parts shows that

$$\int_0^\infty \frac{x^4 e^{2x}}{(e^x - 1)^3} dx = 6 \int_0^\infty \frac{x^2}{e^x - 1} dx + 2 \int_0^\infty \frac{x^3}{e^x - 1} dx . \tag{40}$$

The integrals on the right side of Equation 40 can be expressed in terms of the zeta function $\zeta(s)$, where

$$\zeta(s) = \frac{1}{(s-1)!} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \sum_{n=1}^\infty \frac{1}{n^s} . \tag{41}$$

Therefore

$$\frac{1}{k_r} = \frac{45}{\pi^4} \left(\sum_{n=1}^\infty \frac{1}{n^3} + \sum_{n=1}^\infty \frac{1}{n^4} \right) \tag{42}$$

and

$$k_r = 0.94758593 . \tag{43}$$

Let the desired linear approximation to $B(\tau)$ be denoted by $B^{(0)}(\tau)$. Then, in order to obtain the correct value of the total flux at large optical depths, $B^{(0)}(\tau)$ must satisfy Equation 32 everywhere. Therefore for all τ

$$\frac{dB^{(0)}(\tau)}{d\tau} = \frac{3}{4} k_r F , \tag{44}$$

and $B^{(0)}$, (τ) has the form:

$$B^{(0)}(\tau) = \frac{3}{4} k_r F\tau + B^{(0)}(0) . \tag{45}$$

The constant $B^{(0)}(0)$ is arbitrarily fixed by requiring the function $B^{(0)}(\tau)$ to satisfy the equation

$$2 \int_0^\infty B^{(0)}(t) E_2(t) dt = F . \tag{46}$$

If the problem under discussion were the classical gray atmosphere, Equation 46 would stipulate that $B^{(0)}(\tau)$ yield the prescribed net flux at the surface. This physical interpretation of Equation 46 does not hold for the gray atmosphere of the second kind, as is shown by Equation 26. The result of substituting the expression for $B^{(0)}(\tau)$ from Equation 45 in Equation 46 is

$$B^{(0)}(0) = \frac{1}{2}\left(2-k_r\right) F = 0.526207F . \tag{47}$$

Therefore the linear approximation to the source function is

$$B^{(0)}(\tau) = 0.710689F[\tau + 0.740418] . \tag{48}$$

THE SECOND APPROXIMATION TO THE SOURCE FUNCTION: $B^{(1)}(\tau)$

The second approximation to the source function, $B^{(1)}(\tau)$, is obtained by making two corrections to the linear approximation $B^{(0)}(\tau)$. The two corrections are denoted as $\delta_1 B^{(0)}(\tau)$ and $\delta_2 B^{(0)}(\tau)$, and

$$B^{(1)}(\tau) = B^{(0)}(\tau) + \delta_1 B^{(0)}(\tau) + \delta_2 B^{(0)}(\tau) . \tag{49}$$

The first correction, $\delta_1 B^{(0)}(\tau)$, is defined in terms of the deviation from F of the integrated flux associated with $B^{(0)}(\tau)$. Finding $\delta_1 B^{(0)}(\tau)$ requires an explicit expression for the radiative flux that would exist in the gray atmosphere of the second kind if $B^{(0)}(\tau)$ were the source function.

The expression for the monochromatic optical depth from Equation 23 will now be used to transform Equation 26 into an expression for $F_\nu(\tau)$ which no longer contains the variable τ_ν, i.e.,

$$\begin{aligned} F_\nu(\tau) = {} & 2\int_\tau^\infty \left\{1 - \exp\left[-\frac{h\nu}{kT(t)}\right]\right\} B_\nu(t)\, E_2\left(\int_\tau^t \left\{1 - \exp\left[-\frac{h\nu}{kT(\xi)}\right]\right\} d\xi\right) dt \\ & - 2\int_0^\tau \left\{1 - \exp\left[-\frac{h\nu}{kT(t)}\right]\right\} B_\nu(t)\, E_2\left(\int_t^\tau \left\{1 - \exp\left[-\frac{h\nu}{kT(\xi)}\right]\right\} d\xi\right) dt . \end{aligned} \tag{50}$$

On replacing $B_\nu(t)$ by $2h\nu^3/c^2 \left\{\exp\left[h\nu/kT(t)\right] - 1\right\}^{-1}$, it follows that

$$\begin{aligned} F_\nu(\tau) = {} & \frac{4h\nu^3}{c^2}\int_\tau^\infty \exp\left[-\frac{h\nu}{kT(t)}\right] E_2\left(\int_\tau^t \left\{1 - \exp\left[-\frac{h\nu}{kT(\xi)}\right]\right\} d\xi\right) dt \\ & - \frac{4h\nu^3}{c^2}\int_0^\tau \exp\left[-\frac{h\nu}{kT(t)}\right] E_2\left(\int_t^\tau \left\{1 - \exp\left[-\frac{h\nu}{kT(\xi)}\right]\right\} d\xi\right) dt . \end{aligned} \tag{51}$$

To simplify this expression, write $h\nu/kT(t)$ in the form

$$\frac{h\nu}{kT(t)} = \frac{h\nu}{kT_e}\frac{T_e}{T(t)} , \tag{52}$$

where

$$T_e = \left(\frac{\pi}{\sigma}\right)^{\frac{1}{4}} F^{\frac{1}{4}} \tag{53}$$

is the effective temperature of the atmosphere. Then define the variable y and the function $z(\tau)$ by

$$y = \frac{h\nu}{kT_e} \tag{54}$$

and

$$z(\tau) = \frac{T_e}{T(\tau)} = \left[\frac{F}{B(\tau)}\right]^{\frac{1}{4}} \tag{55}$$

so that

$$\frac{h\nu}{kT(t)} = yz(t) . \tag{56}$$

In terms of these variables

$$F_\nu(\tau) = \frac{30}{\pi^4} F \frac{h}{kT_e} y^3 \int_\tau^\infty e^{-yz(t)} E_2\left\{\int_\tau^t \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt$$

$$- \frac{30}{\pi^4} F \frac{h}{kT_e} y^3 \int_0^\tau e^{-yz(t)} E_2\left\{\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt . \tag{57}$$

With $f_y(\tau)$ defined by

$$f_y(\tau) = \frac{30}{\pi^4} y^3 \int_\tau^\infty e^{-yz(t)} E_2\left\{\int_\tau^t \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt$$

$$- \frac{30}{\pi^4} y^3 \int_0^\tau e^{-yz(t)} E_2\left\{\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt , \tag{58}$$

$$F_\nu(\tau) = \frac{h}{kT_e} F f_y(\tau) . \tag{59}$$

The integrated flux at the level τ is

$$F(\tau) = F \int_0^\infty f_y(\tau)\, dy . \tag{60}$$

The integrated flux that corresponds to the source function $B^{(0)}(\tau)$ will be referred to as $F^{(0)}(\tau)$. It can be found by replacing $B(\tau)$ by $B^{(0)}(\tau)$ in the expression for $f_y(\tau)$ given by Equation 58. Accordingly, set

$$z^{(0)}(\tau) = \left[\frac{F}{B^{(0)}(\tau)}\right]^{\frac{1}{4}} \tag{61}$$

and

$$f_y^{(0)}(\tau) = \frac{30}{\pi^4} y^3 \int_\tau^\infty e^{-yz^{(0)}(t)} E_2 \left\{\int_\tau^t \left[1 - e^{-yz^{(0)}(\xi)}\right] d\xi\right\} dt$$

$$- \frac{30}{\pi^4} y^3 \int_0^\tau e^{-yz^{(0)}(t)} E_2 \left\{\int_t^\tau \left[1 - e^{-yz^{(0)}(\xi)}\right] d\xi\right\} dt . \tag{62}$$

Then

$$F^{(0)}(\tau) = F \int_0^\infty f_y^{(0)}(\tau)\, dy . \tag{63}$$

The extent to which $F^{(0)}(\tau)$ deviates from the prescribed net flux F is measured by the function $F^{(0)}(\tau)$, where

$$\delta F^{(0)}(\tau) = F - F^{(0)}(\tau) = F \left[1 - \int_0^\infty f_y^{(0)}(\tau)\right] dy . \tag{64}$$

The correction $\delta_1 B^{(0)}(\tau)$ is defined in terms of $\delta F^{(0)}(\tau)$ by an equation similar to Equation 44, which defined $B^{(0)}(\tau)$ in terms of F, i.e.,

$$\frac{d}{d\tau} \delta_1 B^{(0)}(\tau) = \frac{3}{4} k_r \delta F^{(0)}(\tau) . \tag{65}$$

The solution to this equation is

$$\delta_1 B^{(0)}(\tau) = \frac{3}{4} k_r \int_0^{\tau} \delta F^{(0)}(t)\, dt + \delta_1 B^{(0)}(0) . \qquad (66)$$

The constant $\delta_1 B^{(0)}(0)$ is chosen in terms of $\delta F^{(0)}(0)$ in the same manner that $B^{(0)}(0)$ is defined in terms of F in Equation 47; i.e.,

$$\delta_1 B^{(0)}(0) = \frac{1}{2} \left(2 - k_r\right) \delta F^{(0)}(0) . \qquad (67)$$

The first correction to $B^{(0)}(\tau)$ is then

$$\delta_1 B^{(0)}(\tau) = \frac{3}{4} k_r \int_0^{\tau} \delta F^{(0)}(t)\, dt + \delta F^{(0)}(0) \left(2 - k_r\right) , \qquad (68)$$

or, with $k_r = 0.94758593$,

$$\delta_1 B^{(0)}(\tau) = 0.710689 \int_0^{\tau} \delta F^{(0)}(t)\, dt + 0.526207\, \delta F^{(0)}(0) . \qquad (69)$$

The second correction to $B^{(0)}(\tau)$, $\delta_2 B^{(0)}(\tau)$ is defined in terms of the derivative of the integrated flux associated with $B^{(0)}(\tau)$. This derivative will be referred to as $dF(0)/d\tau$. It can be expressed in terms of $B^{(0)}(\tau)$ by rewriting Equation 21 so that all the terms are explicit functions of $B^{(0)}(\tau)$.

The monochromatic mean intensity $J_\nu(\tau)$ according to Equation 22 is

$$J_\nu\left[\tau_\nu(\tau)\right] = \frac{1}{2} \int_0^{\infty} B_\nu\left(t_\nu\right) E_1 \left[\left|t_\nu - \tau_\nu(\tau)\right|\right] dt_\nu . \qquad (70)$$

Since

$$\tau_\nu(\tau) = \int_0^{\tau} \left\{1 - \exp\left[-\frac{h\nu}{kT(t)}\right]\right\} dt$$

and

$$B_\nu\left[t_\nu(t)\right] = \frac{2h\nu^3}{c^2}\left\{\exp\left[\frac{h\nu}{kT(t)}\right] - 1\right\}^{-1} ,$$

$$J_\nu(\tau) = \frac{h\nu^3}{c^2}\int_0^\infty \exp\left[-\frac{h\nu}{kT(t)}\right] E_1\left(\left|\int_t^\tau \left\{1 - \exp\left[-\frac{h\nu}{kT(\xi)}\right]\right\} d\xi\right|\right) dt . \tag{71}$$

In terms of the variables y and z of Equations 52 and 53,

$$J_\nu(\tau) = \frac{7.5}{\pi^4} F \frac{h}{kT_e} y^3 \int_0^\infty e^{-yz(t)} E_1\left\{\left|\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right|\right\} dt . \tag{72}$$

Let a quantity $j_y(\tau)$ be defined by

$$j_y(\tau) = \frac{7.5}{\pi^4} y^3 \int_0^\infty e^{-yz(t)} E_1\left\{\left|\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right|\right\} dt . \tag{73}$$

Then

$$J_\nu(\tau) = F \frac{h}{kT_e} j_y(\tau) . \tag{74}$$

The function $B_\nu(\tau)$ can be written as

$$B_\nu(\tau) = \frac{15}{\pi^4} F \frac{h}{kT_e} y^3 \left[e^{yz(\tau)} - 1\right]^{-1} . \tag{75}$$

With $b_y(\tau)$ defined by

$$b_y(\tau) = \frac{15}{\pi^4} y^3 \left[e^{yz(\tau)} - 1\right]^{-1} , \tag{76}$$

$$B_\nu(\tau) = F \frac{h}{kT_e} b_y(\tau) . \tag{77}$$

When these expressions for J_ν and B_ν are substituted into Equation 21, it becomes:

$$\frac{1}{4}\frac{dF(\tau)}{d\tau} = F\int_0^\infty \left[1 - e^{-yz(\tau)}\right]\left[j_y(\tau) - b_y(\tau)\right] dy = 0 . \tag{78}$$

The function $(1/4)\,(dF/d\tau)^{(0)}$ can be found by replacing $B(\tau)$ with $B^{(0)}(\tau)$ in all the terms on the right side of Equation 78. In order to do so, set

$$j_y^{(0)}(\tau) = \frac{75}{\pi^4} y^3 \int_0^\infty e^{-yz^{(0)}(t)} E_1\left\{\left|\int_t^\tau \left[1 - e^{-yz^{(0)}(\xi)}\right] d\xi\right|\right\} dt \tag{79}$$

and

$$b_y^{(0)}(\tau) = \frac{15}{\pi^4} y^3 \left[e^{yz^{(0)}(\tau)} - 1\right]^{-1} . \tag{80}$$

Then

$$\frac{1}{4}\left(\frac{dF}{d\tau}\right)^{(0)} = F\int_0^\infty \left[1 - e^{-yz^{(0)}(\tau)}\right]\left[j_y^{(0)}(\tau) - b_y^{(0)}(\tau)\right] dy . \tag{81}$$

This function is not identically equal to zero because $B^{(0)}(\tau)$ is only an approximation to the source function.

When $B^{(0)}(\tau)$ is corrected by $\delta_2 B^{(0)}(\tau)$, the monochromatic Planck function associated with $B^{(0)}(\tau)$ is also corrected. This monochromatic Planck function is $B_\nu^{(0)}(\tau)$, where

$$B_\nu^{(0)}(\tau) = F\frac{h}{kT_e} b_y^{(0)}(\tau) . \tag{82}$$

The correction to $B_\nu^{(0)}(\tau)$ associated with $\delta_2 B^{(0)}(\tau)$ will be referred to as $\delta_2 B_\nu^{(0)}(\tau)$. It is defined by the following equation:

$$\int_0^\infty \left\{1 - \exp\left[-\frac{h\nu}{kT^{(0)}(\tau)}\right]\right\} \delta_2 B_\nu^{(0)}(\tau)\, d\nu = \frac{1}{4}\left(\frac{dF}{d\tau}\right)^{(0)} , \tag{83}$$

where

$$T^{(0)}(\tau) = \left[\frac{\pi B^{(0)}(\tau)}{\sigma}\right]^{\frac{1}{4}} . \tag{84}$$

Equation 83 can be transformed into an explicit equation for $\delta_2 B^{(0)}(\tau)$ by considering the changes in the temperature distribution within the atmosphere associated with the corrections $\delta_2 B_\nu^{(0)}(\tau)$. If $\delta_2 T^{(0)}(\tau)$ represents the temperature corrections corresponding to $\delta_2 B^{(0)}(\tau)$, then

$$\delta_2 B_\nu^{(0)}(\tau) = B_\nu\left[T^{(0)}(\tau) + \delta_2 T^{(0)}(\tau)\right] - B_\nu\left[T^{(0)}(\tau)\right] . \tag{85}$$

Expanding $B_\nu\left[T^{(0)}(\tau) + \delta_2 T^{(0)}(\tau)\right]$ in a Taylor series of powers of $\delta_2 T^{(0)}(\tau)$ gives the following expression for $\delta_2 B_\nu^{(0)}(\tau)$:

$$\delta_2 B_\nu^{(0)}(\tau) = \sum_{n=1}^{\infty} \frac{1}{n!}\left[\delta_2 T^{(0)}(\tau)\right]^n \left.\frac{\partial^n B_\nu(T)}{\partial T^n}\right]_{T=T_0} . \tag{86}$$

If all powers of $\delta_2 T^{(0)}(\tau)$ except the first are neglected, this expression becomes:

$$\delta_2 B_\nu^{(0)}(\tau) = \delta_2 T^{(0)}(\tau) \left.\frac{\partial B_\nu(T)}{\partial T}\right]_{T=T_0} . \tag{87}$$

The correction function $\delta_2 B^{(0)}(\tau)$ can be expressed as:

$$\delta_2 B^{(0)}(\tau) = B\left[T^{(0)}(\tau) + \delta_2 T^{(0)}(\tau)\right] - B\left[T^{(0)}(\tau)\right] . \tag{88}$$

If $B\left[T^{(0)}(\tau) + \delta_2 T^{(0)}(\tau)\right]$ is expanded in a Taylor series in powers of $\delta_2 T^{(0)}(\tau)$, Equation 88 becomes

$$\delta_2 B^{(0)}(\tau) = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\delta_2 T^{(0)}(\tau)\right)^n \left.\frac{d^n B(T)}{dT^n}\right]_{T=T^{(0)}} . \tag{89}$$

By keeping only the first terms of this series:

$$\delta_2 B^{(0)}(\tau) = \delta_2 T^{(0)}(\tau) \left.\frac{dB}{dT}\right]_{T=T^{(0)}} . \tag{90}$$

It follows from Equations 87 and 90 that

$$\frac{\delta_2 B_\nu^{(0)}(\tau)}{\delta_2 B^{(0)}(\tau)} = \frac{\left.\frac{\partial B_\nu}{\partial T}\right]_{T=T^{(0)}}}{\left.\frac{dB}{dT}\right]_{T=T^{(0)}}} , \tag{91}$$

and therefore

$$\delta_2 B_\nu^{(0)}(\tau) = \left(\frac{\left.\frac{\partial B_\nu}{\partial T}\right]_{T=T^{(0)}}}{\left.\frac{\partial B}{\partial T}\right]_{T=T^{(0)}}} \right) \delta_2 B^{(0)}(\tau) . \tag{92}$$

Substituting this expression for $\delta_2 B_\nu^{(0)}(\tau)$ in Equation 83 gives the following equation for $\delta_2 B^{(0)}(\tau)$:

$$\int_0^\infty \left\{1 - \exp\left[-\frac{h\nu}{kT^{(0)}(\tau)}\right]\right\} \delta_2 B^{(0)}(\tau) \left(\frac{\left.\frac{\partial B_\nu}{\partial T}\right]_{T=T^{(0)}}}{\left.\frac{\partial B}{\partial T}\right]_{T=T^{(0)}}} \right) d\nu = \frac{1}{4}\left(\frac{dF}{d\tau}\right)^{(0)} . \tag{93}$$

With k_G defined by

$$k_G \frac{dB}{dT} = \int_0^\infty \frac{\partial B_\nu}{\partial T}\left[1 - \exp\left(-\frac{h\nu}{kT}\right)\right] d\nu , \tag{94}$$

Equation 93 becomes

$$\delta_2 B^{(0)}(\tau) = \frac{1}{4k_G}\left(\frac{dF}{d\tau}\right)^{(0)} . \tag{95}$$

From Equations 33, 34, and 35 it is found that

$$k_G = \frac{15}{4\pi^4}\frac{h^5}{k^5 T^5}\int_0^\infty \nu^4\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1} d\nu . \tag{96}$$

If the quantity $h\nu/kT$ is replaced by the variable x, the expression for k_G becomes

$$k_G = \frac{15}{4\pi^4}\int_0^\infty \frac{x^4}{e^x - 1}\, dx . \tag{97}$$

This integral can be written in terms of the zeta function $\zeta(s)$ of Equation 41:

$$k_G = \frac{90}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^5} = 0.958057 . \tag{98}$$

The second correction to $B^{(0)}(\tau)$ is therefore

$$\delta_2 B^{(0)}(\tau) = 0.260945 \left(\frac{dF}{d\tau}\right)^{(0)} . \tag{99}$$

The second approximation to the source function is found by adding the correction in Equations 69 and 99 to $B^{(0)}(\tau)$; i.e.,

$$B^{(1)}(\tau) = B^{(0)}(\tau) + 0.710689 \int_0^\tau \delta F^{(0)}(t)\,dt + 0.526207\,\delta F^{(0)}(0) + 0.260945 \left(\frac{dF}{d\tau}\right)^{(0)} . \tag{100}$$

THE METHOD FOR FINDING $B^{(i+1)}(\tau)$ FROM $B^{(i)}(\tau)$ FOR $i \geq 0$

The procedure used to obtain $B^{(1)}(\tau)$ from $B^{(0)}(\tau)$ is the basis of the iteration scheme that gives successively higher approximations to $B(\tau)$. Let us assume that $B^{(i)}(\tau)$ is known. Then for $i \geq 0$,

$$B^{(i+1)}(\tau) = B^{(i)}(\tau) + \delta B^{(i)}(\tau) , \tag{101}$$

where

$$\delta B^{(i)}(\tau) = 0.710689 \int_0^\tau \delta F^{(i)}(t)\,dt + 0.526207\,\delta F^{(i)}(0) + 0.260945 \left(\frac{dF}{d\tau}\right)^{(i)} , \tag{102}$$

$$\delta F^{(i)}(\tau) = F - F^{(i)}(\tau) , \tag{103}$$

$$F^{(i)}(\tau) = F \int_0^\infty f_y^{(i)}(\tau)\,dy , \tag{104}$$

$$f_y^{(i)}(\tau) = \frac{30}{\pi^4} y^3 \int_\tau^\infty e^{-yz^{(i)}(t)} E_2 \left\{ \int_\tau^t \left[1 - e^{-yz^{(i)}(\xi)}\right] d\xi \right\} dt$$

$$- \frac{30}{\pi^4} y^3 \int_0^\tau e^{-yz^{(i)}(t)} E_2 \left\{ \int_t^\tau \left[1 - e^{-yz^{(i)}(\xi)}\right] d\xi \right\} dt \ , \qquad (105)$$

$$z^{(i)}(\tau) = \left[\frac{F}{B^{(i)}(\tau)}\right]^{\frac{1}{4}} , \qquad (106)$$

$$\left(\frac{dF}{d\tau}\right)^{(i)} = 4F \int_0^\infty \left[1 - e^{-yz^{(i)}(\tau)}\right] \left[j_y^{(i)}(\tau) - b_y^{(i)}(\tau)\right] dy \ , \qquad (107)$$

$$j_y^{(i)}(\tau) = \frac{7.5}{\pi^4} y^3 \int_0^\infty e^{-yz^{(i)}(t)} E_1 \left\{ \left| \int_t^\tau \left[1 - e^{-yz^{(i)}(\xi)}\right] d\xi \right| \right\} dt \ , \qquad (108)$$

and

$$b_y^{(i)}(\tau) = \frac{15}{\pi^4} y^3 \left[e^{yz^{(i)}(\tau)} - 1\right]^{-1} \ . \qquad (109)$$

NUMERICAL METHODS USED IN CALCULATING THE SET OF FUNCTIONS $B^{(i)}(\tau)$, $i=0, 1, \cdots, 8$

The functions $B^{(i)}(\tau)$ and $\delta B^{(i)}(\tau)$ depend on the value of the prescribed flux F. This is not true of $B^{(i)}(\tau)/F$ and $\delta B^{(i)}(\tau)/F$. Therefore, $B^{(i)}(\tau)/F$ and $\delta B^{(i)}(\tau)/F$ can be computed without specifying F.

Using the IBM 7094 and 360-75, the functions $B^{(i)}(\tau)/F$ and $\delta B^{(i)}(\tau)/F$, $i = 0, 1, \cdots 8$, have been evaluated at the following 652 values of τ:

$$\tau_1 = 0 , \quad \tau_2 = 10^{-4} , \cdots , \quad \tau_n = 10^{-4+.01(n-2)} , \cdots , \quad \tau_{602} = 100 , \cdots ,$$

$$\tau_{652} \sim 316 \ . \qquad (110)$$

The following additional set of equations was used to compute the monochromatic quantities $f_y^{(i)}(\tau_j)$ and $j_y^{(i)}(\tau_j)$ at $j = 0, \cdots, 652$, and $i = 0, 1, \cdots, 8$:

$$f_y^{(i)}(\tau_j) = \frac{kT_e}{h} F_y^{(i)}(\tau_j)/F \ ; \qquad (111)$$

$$F_y^{(i)}\left(\tau_j\right) = 2\int_{\tau_y^{(i)}(\tau_j)}^{\infty} B_y^{(i)}\left(t_y\right) E_2\left[t_y - \tau_y^{(i)}\left(\tau_j\right)\right] dt_y$$

$$- 2\int_0^{\tau_y^{(i)}(\tau_j)} B_y^{(i)}\left(t_y\right) E_2\left[\tau_y^{(i)}\left(\tau_j\right) - t_y\right] dt_y \ ; \qquad (112)$$

$$\tau_y^{(i)}\left(\tau_j\right) = \int_0^{\tau_j}\left(1 - \exp\left\{- y\left[\frac{F}{B^{(i)}(\tau)}\right]^{\frac{1}{4}}\right\}\right) dt_y \ ; \qquad (113)$$

$$j_y^{(i)}\left(\tau_j\right) = \frac{kT_e}{h} J_y^{(i)}\left(\tau_j\right)\Big/F \ ; \qquad (114)$$

$$J_y^{(i)}\left(\tau_j\right) = \frac{1}{2}\int_0^{\tau_y^{(i)}(\tau_j)} B_y^{(i)}\left(t_y\right) E_1\left[\tau_y^{(i)}\left(\tau_j\right) - t_y\right] dt_y$$

$$+ \frac{1}{2}\int_{\tau_y^{(i)}(\tau_j)}^{\infty} B_y^{(i)}\left(t_y\right) E_1\left[t_y - \tau_y^{(i)}\left(\tau_j\right)\right] dt_y \ ; \qquad (115)$$

$$b_y^{(i)}\left(\tau_j\right) = \frac{kT_e}{h} B_y^{(i)}\left(\tau_y\right)\Big/F \ ; \qquad (116)$$

$$B_y^{(i)}\left(\tau_j\right) = \frac{15}{\pi^4} F \frac{h}{kT_e} y^3 \left(\exp\left\{y\left[\frac{F}{B^{(\tau)}\left(\tau_y\right)}\right]^{\frac{1}{4}} - 1\right\}\right)^{-1} . \qquad (117)$$

The exponential integrals $E_n(x)$ were approximated by a sum of 10 terms by means of the Gauss-Legendre quadrature formula. By definition

$$E_n(x) = \int_0^1 \lambda^{n-2} \exp\left(-\frac{x}{\lambda}\right) d\lambda \ . \qquad (118)$$

In terms of the variable

$$\xi = 2\lambda - 1 \ , \qquad (119)$$

$$E_n(x) = \frac{1}{2^{n-1}}\int_{-1}^{+1} (\xi + 1)^{n-2} \exp\left(-\frac{2x}{\xi + 1}\right) d\xi \ . \qquad (120)$$

The Gauss-Legendre formula states that

$$E_n(x) = \frac{1}{2^{n-1}} \int_{-1}^{+1} (\xi + 1)^{n-2} \exp\left(-\frac{2x}{\xi + 1}\right) d\xi \cong \sum_{i=1}^{10} \frac{1}{2^{n-1}} A_i (\xi_i + 1)^{n-2} \exp\left(-\frac{2x}{\xi_i + 1}\right) . \tag{121}$$

In Equation 121, the 10 constants ξ_i are the roots of the equation

$$P_{10}(x) = 0 , \tag{122}$$

where $P_{10}(x)$ is the 10th order Legendre polynomial, and the 10 constants A_i are

$$A_i = \left[P_{10}'(x_i)\right]^{-2} \left(\frac{2}{1 - x_i^2}\right) , \qquad i = 1, \cdots 10 . \tag{123}$$

In the process of choosing values of y, the y-axis was divided into intervals of length 5. Within each interval, 10 values were chosen, corresponding to the 10 zeroes of the 10th order Legendre polynomial, normalized for that interval. For $i = 0, \cdots, 8$, the largest value of y, $\ell^{(i)}$, was chosen to satisfy the equation:

$$\max_{\tau_j} \left| j_y^{(i)}(\tau_j) - b_y^{(i)}(\tau_j) \right| < 10^{-8} , \tag{124}$$

where

$$y \geq \ell^{(i)} , \qquad j = 0, \cdots, 652 .$$

The monochromatic quantities $f_y^{(i)}(\tau_j)$ and $j_y^{(i)}(\tau_j)$ were computed from Equations 111 through 117, with t_y as the depth variable. The procedure followed was similar to the one used by Lecar (Reference 5), except that the functions $E_n(x)$, $n = 1, 2$, were evaluated by means of Equation 121. It was assumed that $B_y^{(i)}(\tau)$ is a linear function of τ for $\tau > \tau_{652}$ and that

$$B_y^{(i)}(\tau) = B_y^{(i)}(\tau_{652}) + \left[B_y^{(i)}(\tau_{652}) - B_y^{(i)}(\tau_{650})\right] \left(\frac{\tau - \tau_{652}}{\tau_{652} - \tau_{650}}\right) , \tag{125}$$

where

$$\tau > \tau_{652} .$$

The integrations in the variable y, necessary for the evaluation of $\delta F^{(i)}(\tau_j)$ and $dF^{(i)}/d\tau\,(\tau_j)$, were carried out by assuming

$$f_y^{(i)}(\tau_j) = j_y^{(i)}(\tau_j) - b_y^{(i)}(\tau_j) = 0 , \tag{126}$$

where

$$y \geq \ell^{(i)}, \qquad j = 0, \cdots, 652$$

and by using the Gauss-Legendre integration method for $0 \leq y \leq \ell^{(i)}$.

The integrals

$$0.710689 \int_0^{\tau_j} \delta F^{(i)}(t)\,dt$$

were evaluated by means of the trapezoidal rule.

THE CONVERGENCE OF THE SET OF FUNCTIONS $B^{(i)}(\tau)/F$, $i = 0, 1, \cdots, 8$, TO THE SOURCE FUNCTION $B(\tau)/F$

The ideal outcome of the use of this iteration scheme would be the convergence of the set of functions $B^{(i)}(\tau)/F$ to a function $B^{(k)}(\tau)/F$ for which

$$\delta F^{(k)}(\tau_n)/F = 0, \tag{127}$$

$$\left(\frac{dF}{d\tau}\right)^k (\tau_n)/F = 0, \tag{128}$$

$$\delta B^{(k)}(\tau_n)/F = 0, \qquad n = 1, 2, \cdots, 652. \tag{129}$$

The function $B^{(k)}(\tau)/F$ would then be exactly equal to the source function $B(\tau)/F$ at $\tau = \tau_1, \cdots, \tau_{652}$.

An indication of the convergence of the computed sequence of functions $B^{(i)}(\tau)/F$, $i = 0, 1, \cdots, 8$, is provided by Table 1. Values are listed for $\tau_n \leq 100$.

Table 2 gives a more complete indication of the deviation of $F^{(8)}(\tau)/F$ from 1 and the deviation of $dF/d\tau^{(8)}(\tau)/F$ from 0. The numbers in Table 2 have been obtained by interpolation from the values of $F^{(8)}(\tau_n)/F$ and $dF/d\tau^{(8)}(\tau_n)/F$, $n = 1, \cdots, 602$.

Tables 1 and 2 indicate that the function $B^{(8)}(\tau)/F$ comes close to satisfying Equations 127 to 129. Therefore the numerical values computed for $B^{(8)}(\tau)/F$ can be used as a sufficient representation of the source function $B(\tau)/F$.

Table 1

Quantities Illustrating the Convergence of the Sequence of Functions $B^{(i)}(\tau)/F$, $i = 0, 1, \cdots, 8$.

i	The maximum value of $\delta B^{(i)}(\tau)/F$	The value of τ at which $\delta B^{(i)}(\tau)/F$ has a maximum	The maximum value of $\delta F^{(i)}(\tau)/F$	The value of τ at which $\delta F^{(i)}(\tau)/F$ has a maximum	The maximum value of $1/F\,(dF/d\tau)^{(i)}(\tau)$	The value of τ at which $1/F\,(dF/d\tau)^{(i)}(\tau)$ has a maximum
0	-0.0783748	0.000000	-0.020569	0.000000	0.300350	0.000000
1	-0.0050028	0.042658	+0.003884	0.501187	0.017953	0.042658
2	-0.0007000	0.083176	+0.000711	0.512861	0.002573	0.085114
3	+0.0001200	1.122018	+0.000134	0.524807	0.000427	0.117490
4	+0.0000247	1.122018	+0.000030	0.524807	0.000325	99.999985
5	+0.0000051	1.071519	-0.000005	2.041738	0.000325	99.999985
6	+0.0000011	1.096478	-0.000003	1.995262	0.000325	99.999985
7	+0.0000002	1.148154	-0.000003	2.137962	0.000325	99.999985
8	+0.0000000		-0.000003	2.187761	0.000325	99.999985

VALUES OF B(τ)/F AND THE REMAINDER FUNCTION r(τ)

Conventionally, the solution of Milne's integral equation for the classical gray atmosphere is represented as the sum of a linear term and a bounded remainder function, viz.

$$B(\tau) = \frac{3}{4} F\left[\tau + q(\tau)\right] . \tag{130}$$

The variable τ in this solution is defined in terms of the absorption coefficient given by Equation 11. But the same symbol τ is used throughout this paper for another variable defined by Equations 13 and 16. To prevent confusion in what follows, the conventional notation of Equation 130 will here be changed to

$$^{I}B\left(^{I}\tau\right) = \frac{3}{4} F\left[^{I}\tau + q\left(^{I}\tau\right)\right] . \tag{131}$$

Table 2

The Deviation of 1/F $F^{(8)}$ (τ) from 1 and the Deviation of 1/F $\left(dF/d\tau\right)^{(8)}$ (τ) from 0.

τ	1/F $\delta F^{(8)}$ (τ)	1/F $\left(dF/d\tau\right)^{(8)}$ (τ)
0.00	+0.0000005	-0.000000
0.01	+0.0000005	-0.000001
0.05	+0.0000005	-0.000001
0.10	+0.0000005	-0.000001
0.50	-0.0000005	+0.000001
1.00	-0.0000017	+0.000000
1.50	-0.0000024	+0.000003
2.00	-0.0000026	+0.000007
5.00	-0.0000021	+0.000026
10.00	-0.0000018	+0.000052
25.00	-0.0000015	+0.000121
50.00	-0.0000011	+0.000212
75.00	-0.0000008	+0.000278
100.00	-0.0000006	+0.000325

This notation emphasizes the physical distinction between the independent variables $^{I}\tau$ and τ, which are not strictly comparable.

It is convenient to represent the solution for the source function in a gray atmosphere of the second kind in a form similar to Equation 131, writing

$$B(\tau) = 0.710689\ F\left[\tau + r(\tau)\right] . \tag{132}$$

Some numerical values of $B(\tau)$ and of the remainder function $r(\tau)$ are listed in Table 3. Evidently the source function $B(\tau)$ increases monotonically for $\tau \geq 0$. The remainder function $r(\tau)$ is bounded, is always positive, and attains a maximum near $\tau = 1.5$.

If instead of 0.710689 another constant, A, had been chosen on the right-hand side of Equation 127, i.e.,

$$B(\tau) = AF\left[\tau + s(\tau)\right] , \qquad A \neq 0.710689 , \tag{133}$$

then

$$s(\tau) = \left(\frac{0.710689}{A} - 1\right)\tau + \frac{0.710689}{A}\, r(\tau) , \tag{134}$$

and $s(\tau)$ would be unbounded. The preferred constant 0.710689 is identical with the constant appearing in the linear approximation to the correct source function (Equation 48). The reason for this identity is not obvious and bears further investigation.

In order to estimate the accuracy of the solution found for $B(\tau)$ and $r(\tau)$, the functions ${}^{I}B\left({}^{I}\tau\right)$ and $q\left({}^{I}\tau\right)$ were computed in a similar manner, using Lecar's iteration scheme and the numerical approximations of the earlier section of this paper, "Numerical Methods." The values found for $q(0)$ and $q(\infty)$ were correct to six decimal places. The maximum deviation from F in the net flux associated with this computation occurred near $\tau = 0.1$ and was equal to 10^{-6}. The maximum deviation from 0 in the derivative of the net flux occurred near $\tau = 1$ and was equal to 4×10^{-6}. A comparison of the magnitude of these errors with the errors displayed in Table 2 suggests that the values of $B(\tau)$ and $r(\tau)$ listed in Table 3 are probably correct to five decimal places.

THE SURFACE TEMPERATURE

An exact result of the theory of the classical gray atmosphere is the Hopf-Bronstein relation:

$$ {}^{I}q(0) = \frac{1}{\sqrt{3}} . \tag{135} $$

Hence, by Equation 131,

$$ {}^{I}B(0) = \frac{\sqrt{3}}{4} F . \tag{136} $$

Then from Equations 20 and 53 it follows that the boundary temperature is

$$ {}^{I}T(0) = \left[\frac{{}^{I}B(0)}{F}\right]^{\frac{1}{4}} T_e = 0.8112\, T_e . \tag{137} $$

Table 3

Values of the Source Function $B(\tau)/F$ and the Remainder Function $r(\tau)$.

τ	$r(\tau)$	$B(\tau)/F = 0.710689\left[\tau + r(\tau)\right]$
0.00	0.62418	0.443597
0.01	0.63486	0.458293
0.02	0.64253	0.470853
0.03	0.64855	0.482240
0.04	0.65360	0.492937
0.05	0.65803	0.503191
0.06	0.66202	0.513133
0.07	0.66568	0.522837
0.08	0.66906	0.532348
0.09	0.67221	0.541695
0.10	0.67516	0.550898
0.20	0.69700	0.637489
0.30	0.71065	0.718260
0.40	0.71999	0.795964
0.50	0.72667	0.871779
0.60	0.73155	0.946318
0.70	0.73516	1.019951
0.80	0.73783	1.092918
0.90	0.73979	1.165383
1.00	0.74122	1.237466
1.10	0.74222	1.309249
1.20	0.74290	1.380799
1.30	0.74332	1.452164
1.40	0.74353	1.523385
1.50	0.74358	1.594490
1.60	0.74350	1.665502
1.70	0.74332	1.736442
1.80	0.74305	1.807322
1.90	0.74273	1.878158
2.00	0.74235	1.948957
5.00	0.72940	4.071821
10.00	0.72060	7.619014
25.00	0.71459	18.275088
50.00	0.71237	36.040746
75.00	0.71152	53.807382
100.00	0.71107	71.574293

The corresponding relations for the gray atmosphere of the second kind are

$$r(0) = 0.62418 \text{(cf. Table 3)} \tag{138}$$

and

$$B(0) = 0.710689\, r(0)\, F = 0.44360\, F \,, \tag{139}$$

from which it follows that

$$T(0) = 0.8161\, T_e \,. \tag{140}$$

The continued inequality

$$^{I}T(0) < T(0) < T_e \tag{141}$$

calls for some comment.

The effective temperature can be regarded as that of an *isothermal* blackbody, of infinite optical thickness, whose surface emits the same flux as emerges from the two atmospheres here compared. The inequalities stated suggest that the temperature stratification in a gray atmosphere of the second kind is closer to *isothermy* than is that in a classical gray atmosphere, or, in other words, that the average temperature gradient throughout the layers substantially contributing to the escaping radiation is smaller than in the classical case. A precise comparison of the two temperature gradients is impossible because the optical depths τ and $^{I}\tau$ differ with respect to their physical definition. Consequently, the scales of τ and $^{I}\tau$ cannot unambiguously be related to a common geometric scale. In these circumstances a somewhat arbitrary, though plausible, assumption has to be made. Let τ^* and $^{I}\tau^*$ be the optical depths at which the respective source functions equal the flux constant. Then

$$\tau^* = 0.6728 \,, \qquad B(\tau^*) = F \,, \tag{142}$$

and

$$^{I}\tau^* = 0.6454 \,, \qquad ^{I}B(^{I}\tau^*) = F \,. \tag{143}$$

It is assumed that these levels are approximately equivalent physically, in the sense that the average temperature gradients between the starred levels and the surfaces lend themselves to a meaningful comparison. Now these gradients are

$$\left(\frac{\pi}{\sigma}\right)^{\frac{1}{4}} \left\{ \frac{\left[B(\tau^*)\right]^{\frac{1}{4}} - \left[B(0)\right]^{\frac{1}{4}}}{\tau^*} \right\} = 4.1937 \tag{144}$$

and

$$\left(\frac{\pi}{\sigma}\right)^{\frac{1}{4}} \left\{ \frac{\left[{}^{I}B\left({}^{I}\tau^{*}\right)\right]^{\frac{1}{4}} - \left[{}^{I}B(0)\right]^{\frac{1}{4}}}{{}^{I}\tau^{*}} \right\} = 4.4882 \ . \tag{145}$$

The gradient of Equation 144 is indeed smaller than the classical gradient (Equation 145).

THE MONOCHROMATIC FLUXES: $F_\nu(\tau)$

The frequency distribution of the radiant fluxes at any level in the atmosphere is given by the function $\pi F_\nu (\tau)\, d\nu$ [erg/cm²-sec] appearing in Equation 26. According to Equation 59, the normalize monochromatic net flux

$$f_y(\tau) = \frac{F_\nu(\tau)}{F} \frac{kT_e}{h} \tag{146}$$

is a function of τ only. Some values of $f_y(\tau)$ are given in Table 4. They were found by interpolation from the more extensive set used in the computation of $B(\tau)/F$.

The frequency distribution of the flux changes progressively as the radiation flows towards the surface. The monochromatic flux at high frequencies decreases, while the monochromatic flux at low frequencies increases. The frequency at which the monochromatic flux is a maximum shifts towards the red end of the spectrum, and the maximum increases as the radiation approaches the surface. This pattern of redistribution, or flux reddening, is illustrated in Figure 1, which contains graphs of the function $f_y(\tau)$, evaluated at τ = 4, 3, 2, 1, 1/2, and 0.

The monochromatic flux spectrum at the surface of a classical gray atmosphere with net radiant flux πF will be represented by the symbol ${}^{I}F_\nu(0)$ and written in the form

$$F_\nu(0) = \frac{h}{kT_e} F\, {}^{I}f_y \ , \tag{147}$$

where

$${}^{I}f_y = \frac{30}{\pi^4} y^3 \int_0^\infty \left\{ \exp\left[y\, {}^{I}z\left({}^{I}\tau\right)\right] - 1 \right\}^{-1} E_2\left({}^{I}\tau\right) d\, {}^{I}\tau \tag{148}$$

and

$${}^{I}z\left({}^{I}\tau\right) = \frac{F}{{}^{I}B\left({}^{I}\tau\right)} \ . \tag{149}$$

Table 4

The Function $f_y(\tau)$.

	y = 1	2	3	4	5	6	8	10	12	14
0.0	0.093518	0.189391	0.209125	0.176314	0.127426	0.083523	0.030094	0.009506	0.002808	0.000802
0.001	0.093461	0.189284	0.209068	0.176324	0.127472	0.083574	0.030123	0.009517	0.002812	0.000803
0.01	0.092952	0.188334	0.208561	0.176412	0.127874	0.084024	0.030381	0.009615	0.002843	0.000812
0.1	0.088326	0.179625	0.203633	0.176903	0.131428	0.088197	0.032893	0.010606	0.003167	0.000908
0.2	0.083827	0.171055	0.198365	0.176894	0.134691	0.092351	0.035604	0.011733	0.003550	0.001025
0.3	0.079822	0.163354	0.193302	0.176474	0.137402	0.096096	0.038238	0.012884	0.003956	0.001153
0.4	0.076212	0.156362	0.188445	0.175749	0.139660	0.099486	0.040796	0.014052	0.004382	0.001290
0.5	0.072930	0.149972	0.183794	0.174790	0.141535	0.102560	0.043274	0.015233	0.004827	0.001437
0.6	0.069927	0.144108	0.179345	0.173652	0.143081	0.105347	0.045670	0.016422	0.005289	0.001593
0.7	0.067167	0.138705	0.175094	0.172374	0.144341	0.107872	0.047983	0.017614	0.005766	0.001759
0.8	0.064621	0.133713	0.171035	0.170989	0.145353	0.110160	0.050213	0.018806	0.006256	0.001933
0.9	0.062263	0.129088	0.167159	0.169524	0.146148	0.112230	0.052358	0.019993	0.006759	0.002116
1.0	0.060073	0.124790	0.163459	0.167999	0.146754	0.114102	0.054421	0.021174	0.007271	0.002306
1.2	0.056134	0.117064	0.156557	0.164835	0.147484	0.117311	0.058304	0.023506	0.008323	0.002709
1.4	0.052691	0.110322	0.150264	0.161598	0.147695	0.119905	0.061873	0.025787	0.009400	0.003138
1.6	0.049659	0.104397	0.144516	0.158358	0.147509	0.121984	0.065148	0.028004	0.010494	0.003591
1.8	0.046972	0.099158	0.139256	0.155159	0.147014	0.123628	0.068147	0.030150	0.011598	0.004063
2.0	0.044577	0.094498	0.134432	0.152030	0.146284	0.124907	0.070892	0.032221	0.012706	0.004553
3.0	0.035737	0.077378	0.115405	0.137887	0.140685	0.127527	0.081425	0.041393	0.018135	0.007173
4.0	0.030144	0.066535	0.102148	0.126352	0.134046	0.126664	0.088089	0.048723	0.023160	0.009908

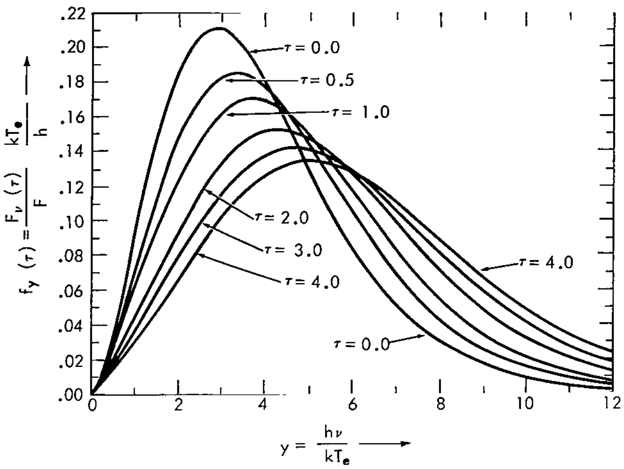

Figure 1—The frequency distribution of the net flux of radiation at various levels in an Einstein-gray atmosphere.

Some numerical values of the function ${}^{I}f_y$, computed by the same methods used to find $f_y(0)$, are listed in Table 5. These values were obtained from two independent determinations of the function ${}^{I}z\,({}^{I}\tau)$, one based on the discrete ordinate method (using 96 ordinates), the other based on the iterative procedure.

The monochromatic flux spectrum at the surface of the blackbody atmosphere with escaping total flux πF will be denoted by the symbol ${}^{B.B.}F_\nu\,(F)$ and written in the form

$$ {}^{B.B.}F_\nu\,(F) = \frac{h}{kT_e}\, F\, {}^{B.B.}f_y \,. \tag{150}$$

where

$$ {}^{B.B.}f_y = \frac{15}{\pi^4}\,\frac{y^3}{e^y - 1}\,. \tag{151}$$

Values of this function also are listed in Table 5.

Table 5

Surface Monochromatic Fluxes.

$y = \frac{h\nu}{kT_e}$	$f_y(0) = \left[\frac{F_\nu(0)}{F}\right]\left(\frac{kT_e}{h}\right)$	${}^{I}f_y = \left[\frac{{}^{I}F_\nu(0)}{F}\right]\left(\frac{kT_e}{h}\right)$	${}^{B.B.}f_y = \left[\frac{{}^{B.B.}F_\nu(F)}{F}\right]\left(\frac{kT_e}{h}\right)$
0	0.000000	0.000000	0.000000
1	0.093518	0.086447	0.089618
2	0.189391	0.184379	0.192817
3	0.209125	0.208345	0.217847
4	0.176314	0.178185	0.183875
5	0.127426	0.130186	0.130577
6	0.083523	0.086158	0.082653
8	0.030094	0.031632	0.026458
10	0.009506	0.010190	0.006991
12	0.002808	0.003072	0.001635
14	0.000802	0.000895	0.000351

Table 6 lists the maxima of these surface monochromatic fluxes and the values of y at which these maxima are attained.

The ratio of the fluxes escaping from the gray atmosphere of the second kind to the fluxes escaping from the isothermal blackbody atmosphere, expressed as differences in the scale of astronomical magnitudes, is

$$\Delta^{II} m_\nu = -2.5 \log \frac{F_\nu(0)}{{}^{B.B.}F_\nu(F)} . \qquad (152)$$

If the dimensionless parameter $y = h\nu/kT_e$ is used instead of the frequency parameter ν, as the dependent variable, the magnitude differerences become

$$\Delta^{II} m_y = -2.5 \log \frac{f_y}{{}^{B.B.}f_y} . \qquad (153)$$

A graph of $\Delta^{II} m_y$ appears in Figure 2.

Similarly, the magnitude differences between a classical gray atmosphere and an isothermal blackbody atmosphere with the same surface flux are

$$\Delta^{I} m_\nu = -2.5 \log \frac{{}^{I}F_\nu(0)}{{}^{B.B.}F_\nu(F)} \qquad (154)$$

or

$$\Delta^{I} m_y = -2.5 \log \frac{{}^{I}f_y}{{}^{B.B.}f_y} . \qquad (155)$$

Figure 2 contains a graph of $\Delta^I m_y$.

Table 6

Maximum Surface Fluxes.

y max	$f_{y\,max}$	${}^{I}f_{y\,max}$	${}^{B.B.}f_{y\,max}$
2.77	0.210698		
2.83		0.209158	
2.82			0.218886

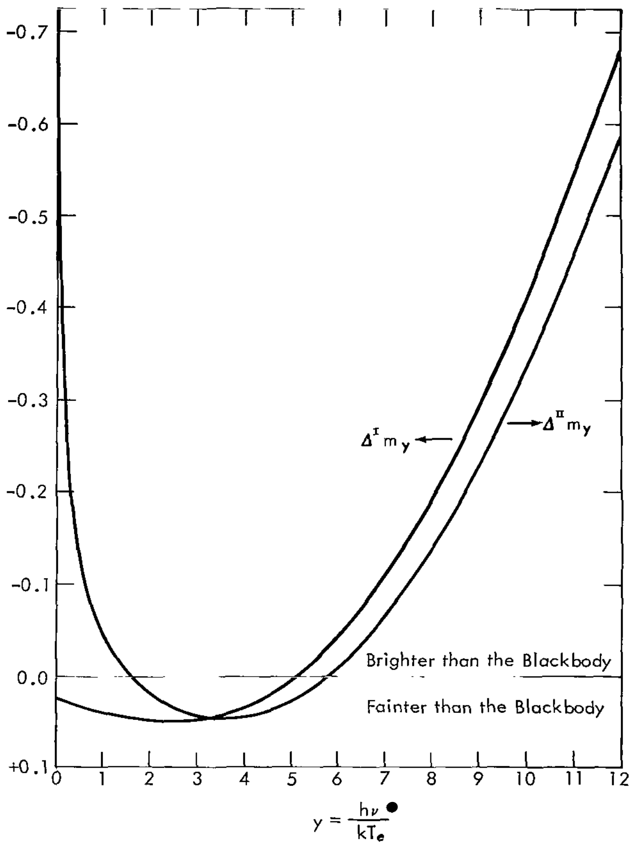

Figure 2—The monochromatic magnitude differences between an Einstein-gray atmosphere and a blackbody at the effective temperature T_e are given by the graph labeled $\Delta^{II} m_y$ ($\lim_{y\to 0} \Delta^{II} m_y = \infty$). The monochromatic magnitude differences between a classical gray atmosphere and a blackbody at the effective temperature T_e are given by the graph labeled $\Delta^I m_y$.

The graph of $\Delta^I m_y$ shows that the classical gray atmosphere is brighter than the blackbody for $y \geq 5.08$ and fainter than the blackbody for $y < 5.08$. The radiation escaping from the deeper layers of an atmosphere is coming from levels at temperatures higher than the surface temperature. Since the Planck function increases more rapidly in the ultraviolet than in the infrared, the spectrum

escaping from the classical gray atmosphere must be bluer than that of the blackbody, since the absorption is "gray." However, because the integrated flux from these two atmospheres is the same, the blackbody is brighter at the red end of the spectrum. A detailed discussion of this phenomenon has been given by Wildt (References 6 and 7).

The graph of $\Delta^{II} m_y$ shows that the gray atmosphere of the second kind is brighter than the blackbody for $0 \leq y \leq 1.59$ and for $y > 5.75$. It is brighter than the blackbody in the ultraviolet for the same reason that the classical gray atmosphere is brighter there. Its excess brightness in the far infrared must be due to the high transparency of the gray atmosphere of the second kind at small values of ν which results from the Rosseland factor

$$\left\{1 - \exp\left[-\frac{h\nu}{kT(\tau)}\right]\right\} .$$

This quantity increases with depth and evidently makes for such transparency that the emergent flux spectrum is greater than that of the blackbody. Since the integrated flux from the gray atmosphere of the second kind is the same as that from the blackbody, there must be an intermediate spectral region where the blackbody is brighter.

THE LIMB DARKENING IN TOTAL RADIATION: $I(0, \mu)/I(0, 1)$

The specific intensity of the total radiation escaping from the surface of a gray atmosphere of the second kind with effective temperature T_e' at an angle $\theta\left(\cos^{-1}\mu\right)$ to the normal, is

$$I(0, \mu) = \int_0^\infty I_\nu (0, \mu)\, d\nu , \qquad 0 \leq \mu \leq 1 . \tag{156}$$

The limb-darkening ratio, in total radiation, of a gray atmosphere of the second kind is

$$\frac{I(0, \mu)}{I(0, 1)} = \frac{\int_0^\infty I_\nu (0, \mu)\, d\nu}{\int_0^\infty I_\nu (0, 1)\, d\nu} , \qquad 0 \leq \mu \leq 1 . \tag{157}$$

As is well known, the function $I_\nu (0, \mu)$ is the Laplace transform of the monochromatic source function, i.e.,

$$I_\nu (0, \mu) = \frac{1}{\mu} \int_0^\infty B_\nu \left[\tau_\nu (\tau)\right] \exp\left(-\frac{\tau_\nu}{\mu}\right) d\tau_\nu . \tag{158}$$

Since

$$\tau_\nu(\tau) = \int_0^\tau \left\{1 - \exp\left[-\frac{h\nu}{kT(t)}\right]\right\} dt$$

and

$$B_\nu\left[\tau_\nu(\tau)\right] = \frac{2h\nu^3}{c^2}\left\{\exp\left[\frac{h\nu}{kT(\tau)}\right] - 1\right\}^{-1},$$

$$I_\nu(0, \mu) = \frac{2h\nu^3}{c^2\mu}\int_0^\infty \exp\left(-\frac{h\nu}{kT(\tau)} - \frac{1}{\mu}\int_0^\tau \left\{1 - \exp\left[-\frac{h\nu}{kT(t)}\right]\right\} dt\right) d\tau . \tag{159}$$

If ν is replaced by $\left(kT_e/h\right) y$, $h\nu/kT(\tau)$ is replaced by $yz(\tau)$, and $\left(2\pi^4 k^4 T_e^4\right)/\left(15c^2 h^3\right)$ is replaced by F,

$$I(0, \mu) = \int_0^\infty I_\nu(0, \mu)\, d\nu = \frac{15}{\pi^4\mu} F \int_0^\infty \int_0^\infty y^3 \exp\left(-yz(\tau) - \frac{1}{\mu}\int_0^\tau \left\{1 - \exp\left[-yz(t)\right]\right\} dt\right) d\tau\, dy \tag{160}$$

and

$$\frac{I(0, \mu)}{I(0, 1)} = \frac{\int_0^\infty \int_0^\infty y^3 \exp\left(-yz(\tau) - \frac{1}{\mu}\int_0^\tau \left\{1 - \exp\left[-yz(t)\right]\right\} dt\right) d\tau\, dy}{\mu \int_0^\infty \int_0^\infty y^3 \exp\left(-yz(\tau) - \int_0^\tau \left\{1 - \exp\left[-yz(t)\right]\right\} dt\right) d\tau\, dy} . \tag{161}$$

This limb-darkening ratio was computed for 12 values of μ. The results are listed in Table 7.

The limb darkening of a classical gray atmosphere will be denoted by ${}^{I}I(0, \mu)/{}^{I}I(0, 1)$. Then

$$\frac{{}^{I}I(0, \mu)}{{}^{I}I(0, 1)} = \frac{\int_0^\infty {}^{I}I_\nu(0, \mu)\, d\nu}{\int_0^\infty {}^{I}I_\nu(0, 1)\, d\nu}, \qquad 0 \leq \mu \leq 1 , \tag{162}$$

where

$$ {}^{I}I_{\nu}(0, \mu) = \frac{1}{\mu} \int_0^{\infty} {}^{I}B_{\nu}\left({}^{I}\tau\right) \exp\left(- \frac{{}^{I}\tau}{\mu}\right) d^{I}\tau , \tag{163} $$

$$ {}^{I}B_{\nu} \; {}^{I}\tau = \frac{2h\nu^3}{c^2} \left\{ \exp\left[\frac{h\nu}{k \, {}^{I}T\left({}^{I}\tau\right)}\right] - 1 \right\}^{-1} , \tag{164} $$

and ${}^{I}T\left({}^{I}\tau\right)$ is the temperature at the optical depth ${}^{I}\tau$ in the atmosphere.

If the variable ν is replaced by $\left(kT_e/h\right) y$ and the function $h\nu/k \, {}^{I}T\left({}^{I}\tau\right)$ is replaced by $y \, {}^{I}z\left({}^{I}\tau\right)$ the limb darkening ratio becomes

$$ \frac{{}^{I}I(0, \mu)}{{}^{I}I(0, 1)} = \frac{\int_0^{\infty} \int_0^{\infty} y^3 \left\{ \exp\left[y \, {}^{I}z\left({}^{I}\tau\right)\right] - 1 \right\}^{-1} \exp\left(- \frac{{}^{I}\tau}{\mu}\right) d\, {}^{I}\tau \, dy}{\mu \int_0^{\infty} \int_0^{\infty} y^3 \left\{ \exp\left[y \, {}^{I}z\left({}^{I}\tau\right)\right] - 1 \right\}^{-1} \exp\left(- {}^{I}\tau\right) d\, {}^{I}\tau \;\; d\, {}^{I}\tau \, dy} \; . \tag{165} $$

This ratio was computed at the 12 values of μ listed in Table 7.

The limb darkening of the Einstein-gray atmosphere is slightly less extreme than the limb darkening characteristic of the classical gray atmosphere.

Table 7

Limb Darkening in Total Radiation.

$\mu = \cos\theta$	(1) $\frac{I(0, \mu)}{I(0, 1)}$	(2) $\frac{{}^{I}I(0, \mu)}{{}^{I}I(0, 1)}$
1.00	1.00000	1.00000
0.90	0.94023	0.93905
0.80	0.88023	0.87788
0.70	0.81993	0.81641
0.60	0.75923	0.75457
0.50	0.69800	0.69220
0.40	0.63599	0.62909
0.30	0.57280	0.56487
0.20	0.50766	0.49878
0.10	0.43864	0.42897
0.05	0.40082	0.39087
0.00	0.35402	0.34390

(1) Einstein-gray atmosphere.
(2) Classical gray atmosphere.
Values are based on K. Grossman's computations of the source function of the classical gray atmosphere (using 96 discrete ordinates).

THE $\mathcal{H}(\mu)$ FUNCTION

The angular dependence of the radiation intensity at the surface of the gray atmosphere of the second kind can be written in terms of a dimensionless function $\mathcal{H}(\mu)$, defined by

$$ I(0, \mu) = B(0) \mathcal{H}(\mu) . \tag{166} $$

This function is the analogue of the function $H(\mu)$ introduced by Chandrasekhar into the theory of the classical gray atmosphere.

From Equation 158 it follows that

$$ I_{\nu}(0, 0) = B_{\nu}(0) ; \tag{167} $$

i.e., the specific intensity of monochromatic radiation emerging tangentially from the surface is equal to the value of the monochromatic source function at $\tau = 0$. Therefore

$$I(0,0) = B(0) , \tag{168}$$

and

$$\mathcal{H}(0) = 1 . \tag{169}$$

Table 8 lists some values of $\mathcal{H}(\mu)$ obtained by computing $I(0, \mu)$ from Equation 160 and by using the value of $B(0)$ given by Equation 139. The computed value of $\mathcal{H}(0)$ differs from the theoretical value by 9×10^{-6}. Table 8 also lists comparable values of $H(\mu)$ taken from Placzek (Reference 8).

Table 8

The $\mathcal{H}(\mu)$-Function.

μ	$\mathcal{H}(\mu)$	$H(\mu)$
1.00	2.824759	2.90781
0.90	2.655919	2.73059
0.80	2.486429	2.55270
0.70	2.316097	2.37398
0.60	2.144650	2.19413
0.50	1.971673	2.01278
0.40	1.796510	1.82928
0.30	1.618032	1.64252
0.20	1.434029	1.45035
0.10	1.239040	1.24735
0.05	1.132214	1.13658
0.00	1.000009	1.00000

PHOTON DIFFUSION

A measure of the progressive reddening that the flux undergoes as it approaches the surface is the ratio of the local rate of emission of photons to the local rate of absorption of photons, both expressed in units of the dimension $[\mathrm{cm}^{-3}\ \mathrm{sec}^{-1}]$. This ratio will be denoted as ${}^{II}m(\tau)$, while the corresponding ratio for the classical gray atmosphere will here be denoted as ${}^{I}m\,({}^{I}\tau)$; the latter quantity is the function $m(\tau)$ of Wildt (Reference 6).

In the gray atmosphere of the second kind, the number of photons emitted per cubic centimeter per second at the level τ is

$$\alpha(\tau) \int_0^\infty \left\{1 - \exp\left[-\frac{h\nu}{kT(\tau)}\right]\right\} \frac{B_\nu(\tau)}{h\nu}\, d\nu = \frac{2k^3 T_e^3}{c^2 h^3}\, \alpha(\tau) \int_0^\infty y^2 e^{-yz(\tau)}\, dy ,$$

and the corresponding number of photons absorbed is

$$\alpha(\tau) \int_0^\infty \left\{1 - \exp\left[-\frac{h\nu}{kT(\tau)}\right]\right\} \frac{J_\nu(\tau)}{h\nu} .$$

An expression for $J_\nu(\tau)$ is given in Equation 71. In terms of the variable y and the function $z(\tau)$, the number of photons absorbed is

$$\frac{k^3 T_e^3}{c^2 h^3}\, \alpha(\tau) \int_0^\infty y^3 \left[1 - e^{-yz(\tau)}\right] \left(\int_0^\infty e^{-yz(t)} E_1\left\{\left|\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right|\right\} dt\right) dy .$$

The function ${}^{II}m(\tau)$ is then

$$
{}^{II}m(\tau) = \frac{2\int_0^\infty y^2\, e^{-yz(\tau)}\, dy}{\int_0^\infty y^3\left[1-e^{-yz(\tau)}\right]\left(\int_0^\infty e^{-yz(t)}\, E_1\left\{\left|\int_t^\tau \left[1-e^{-yz(\xi)}\right] d\xi\right|\right\} dt\right) dy} \cdot \tag{170}
$$

Some values of ${}^{II}m(\tau)$ are listed in Table 9.

Table 9

Photon Multiplication Factor and Normalized Divergence of the Photon Net Flux.

τ	${}^{II}m(\tau)$	${}^{II}\Delta(\tau)$	${}^{I}\tau$	${}^{I}m({}^{I}\tau)$	${}^{I}\Delta({}^{I}\tau)$
0.00	1.16907	.144617	0.00	1.21884	0.179546
0.01	1.16007	.137984	0.01	1.20407	0.169480
0.02	1.15306	.132741	0.02	1.19361	0.162207
0.03	1.14723	.128338	0.03	1.18485	0.156014
0.04	1.14218	.124478	0.04	1.17720	0.150525
0.05	1.13765	.120996	0.05	1.17035	0.145555
0.10	1.11967	.106877	0.10	1.14377	0.125697
0.20	1.09549	.087168	0.20	1.10984	0.098969
0.30	1.07924	.073424	0.30	1.08797	0.080858
0.40	1.06732	.063075	0.40	1.07246	0.067565
0.50	1.05812	.054932	0.50	1.06085	0.057358
1.00	1.03212	.031122	1.00	1.03006	0.029187
1.50	1.02027	.019868	1.50	1.01742	0.017120
2.00	1.01381	.013620	2.00	1.01110	0.010976
2.50	1.00992	.009821	2.50	1.00757	0.007514
3.00	1.00742	.007362	3.00	1.00545	0.005416
3.50	1.00573	.005693	3.50	1.00409	0.004070
4.00	1.00454	.004517	4.00	1.00318	0.003165
4.50	1.00368	.003662	4.50	1.00254	0.002531
5.00	1.00303	.003023	5.00	1.00208	0.002071
10.00	1.00082	.000816	10.00	1.00056	0.000559
100.00	1.00001	.000013	100.00	1.00001	0.000008

From the photon multiplication factor ${}^{\mathrm{II}}m(\tau)$ it is possible to compute

$$ {}^{\mathrm{II}}\Delta(\tau) = 1 - \frac{1}{{}^{\mathrm{II}}m(\tau)} \ . \tag{171} $$

This is the divergence of the photon net flux divided by the local rate of emission of photons, a dimensionless quantity called by Wildt (Reference 6) the normalized divergence of the photon net flux. The corresponding ratio for the classical gray atmosphere will here be denoted as ${}^{\mathrm{I}}\Delta\left({}^{\mathrm{I}}\tau\right)$. It is the function $\Delta(\tau)$ of Wildt (Reference 6). Some values of ${}^{\mathrm{II}}\Delta(\tau)$ and ${}^{\mathrm{I}}\Delta\left({}^{\mathrm{I}}\tau\right)$ are given in Table 9.

The photon flux emerging from the surface of the gray atmosphere of the second kind will be denoted by $\pi\, {}^{\mathrm{II}}D(0)$. Clearly,

$$ {}^{\mathrm{II}}D(0) = 4 \int_0^\infty \int_0^\infty \frac{\nu^2}{c^2} \exp\left[-\frac{h\nu}{kT(\tau)}\right] E_2\left(\int_0^\tau \left\{1 - \exp\left[-\frac{h\nu}{kT(t)}\right]\right\} dt\right) d\tau\, d\nu $$

$$ = 4\,\frac{k^3 T_e^3}{h^3 c^2} \int_0^\infty \int_0^\infty y^2 e^{-yz(\tau)} E_2 \left\{\int_0^\tau \left[1 - e^{-yz(t)}\right] dt\right\} d\tau\, dy \ . \tag{172} $$

The photon flux at the surface of an isothermal blackbody, at the temperature T_e throughout, can be denoted by $\pi\,{}^{\mathrm{B.B.}}D(F)$, where

$$ {}^{\mathrm{B.B.}}D(F) = 2 \int_0^\infty \frac{\nu^2}{c^2} \left[\exp\left(\frac{h\nu}{kT_e}\right) - 1\right]^{-1} d\nu $$

$$ = \frac{2k^3 T_e^3}{c^2 h^3} \int_0^\infty \frac{y^2}{e^y - 1}\, dy \ . \tag{173} $$

Then

$$ \frac{{}^{\mathrm{II}}D(0)}{{}^{\mathrm{B.B.}}D(F)} = \frac{\displaystyle\int_0^\infty \int_0^\infty y^2 e^{-yz(\tau)} E_2 \left\{\int_0^\tau \left[1 - e^{-yz(t)}\right] dt\right\} d\tau\, dy}{\zeta(3)} \ . \tag{174} $$

Computations indicate

$$ \frac{{}^{\mathrm{II}}D(0)}{{}^{\mathrm{B.B.}}D(F)} = \frac{1.214926}{1.202057} = 1.010706 \ . \tag{175} $$

Therefore the radiation from the gray atmosphere of the second kind is in excess of photons by 1.1 percent when compared with the radiation from the blackbody of the same effective temperature.

THE RADIATIVE STRESS TENSOR

Let ${}^{\text{II}}\Pi_{1,\nu}(\tau)$, ${}^{\text{II}}\Pi_{2,\nu}(\tau)$ and ${}^{\text{II}}\Pi_{3,\nu}(\tau)$ denote the principal values of the monochromatic radiative stress tensor associated with a three-dimensional rectangular coordinate system originating at the level τ, the first axis of which points in the negative τ direction. The integrated radiative stress tensor has the components

$$ {}^{\text{II}}\Pi_1(\tau) = \int_0^\infty {}^{\text{II}}\Pi_{1,\nu}(\tau)\, d\nu , \qquad {}^{\text{II}}\Pi_2(\tau) = \int_0^\infty {}^{\text{II}}\Pi_{2,\nu}(\tau)\, d\nu , $$

and

$$ {}^{\text{II}}\Pi_3(\tau) = \int_0^\infty {}^{\text{II}}\Pi_{3,\nu}(\tau)\, d\nu . $$

It is the analogue of the radiative stress tensor of the classical gray atmosphere, which has principal values $\Pi_1({}^{\text{I}}\tau)$, $\Pi_2({}^{\text{I}}\tau)$, and $\Pi_3({}^{\text{I}}\tau)$ (Reference 7). The components ${}^{\text{II}}\Pi_1(\tau)$, ${}^{\text{II}}\Pi_2(\tau)$ and ${}^{\text{II}}\Pi_3(\tau)$ are related to the mean radiative stress at the level τ, $\overline{{}^{\text{II}}P(\tau)}$, by

$$ {}^{\text{II}}\Pi_1(\tau) + {}^{\text{II}}\Pi_2(\tau) + {}^{\text{II}}\Pi_3(\tau) = 3\,\overline{{}^{\text{II}}P(\tau)} . \tag{176} $$

The mean radiative stress at the level ${}^{\text{I}}\tau$ in the classical gray atmosphere is $\overline{P({}^{\text{I}}\tau)}$ (Reference 7).

The first principal value of the monochromatic stress tensor is

$$ {}^{\text{II}}\Pi_{1,\nu}(\tau) = \frac{2\pi}{c} \int_{-1}^{+1} I_\nu(\tau, \mu)\, \mu^2\, d\mu . \tag{177} $$

The corresponding component of the integrated stress tensor is therefore

$$ {}^{\text{II}}\Pi_1(\tau) = \frac{2\pi}{c} \int_0^\infty \int_{-1}^{+1} I_\nu(\tau, \mu)\, \mu^2\, d\mu\, d\nu . \tag{178} $$

Rotational symmetry about the τ-axis implies that ${}^{\text{II}}\Pi_2(\tau) = {}^{\text{II}}\Pi_3(\tau)$. Then from Equation 176 it follows that

$$ {}^{\text{II}}\Pi_2(\tau) = {}^{\text{II}}\Pi_3(\tau) = \frac{1}{2}\left[3\,\overline{{}^{\text{II}}P(\tau)} - {}^{\text{II}}\Pi_1(\tau)\right] . \tag{179} $$

Since the mean radiative stress at the level τ is

$$\overline{{}^{\text{II}}P(\tau)} = \frac{2\pi}{3c} \int_0^\infty \int_{-1}^{+1} I_\nu (\tau, \mu)\, d\mu\, d\nu \ , \tag{180}$$

$$^{\text{II}}\Pi_2 (\tau) = {}^{\text{II}}\Pi_3 (\tau) = \frac{\pi}{c} \int_0^\infty \int_{-1}^{+1} I_\nu (\tau, \mu)\, d\mu\, d\nu - \frac{\pi}{c} \int_0^\infty \int_{-1}^{+1} I_\nu (\tau, \mu)\, \mu^2\, d\nu \ . \tag{181}$$

The monochromatic principal values must then be

$$^{\text{II}}\Pi_{2,\nu} (\tau) = \Pi_{3,\nu} (\tau) = \frac{\pi}{c} \int_{-1}^{+1} I_\nu (\tau, \mu)\, d\mu - \frac{\pi}{c} \int_{-1}^{+1} I_\nu (\tau, \mu)\, \mu^2\, d\nu \ . \tag{182}$$

The function $^{\text{II}}\Pi_{1,\nu} (\tau)$ can be computed from the source function $B(\tau)$. From the equation of transfer relevant to the Einstein-gray atmosphere it follows that

$$^{\text{II}}\Pi_{1,\nu} \left[\tau_\nu (\tau)\right] = \frac{2\pi}{c} \int_{\tau_\nu (\tau)}^\infty B_\nu \left[t_\nu (t)\right] E_3 \left[t_\nu - \tau_\nu (\tau)\right] dt_\nu$$

$$+ \frac{2\pi}{c} \int_0^{\tau_\nu (\tau)} B_\nu \left[t_\nu (t)\right] E_3 \left[\tau_\nu (\tau) - t_\nu\right] dt_\nu \ . \tag{183}$$

Since

$$\tau_\nu (\tau) = \int_0^\infty \left\{1 - \exp \left[- \frac{h\nu}{kT(t)}\right]\right\} dt \ ,$$

$$^{\text{II}}\Pi_{1,\nu} (\tau) = \frac{2\pi}{c} \int_\tau^\infty \left\{1 - \exp \left[- \frac{h\nu}{kT(\tau)}\right] B_\nu (t)\right\} E_3\left(\int_\tau^t \left\{1 - \exp \left[- \frac{h\nu}{kT(\xi)}\right]\right\} d\xi\right) dt$$

$$+ \frac{2\pi}{c} \int_0^\tau \left\{1 - \exp \left[- \frac{h\nu}{kT(\tau)}\right] B_\nu (t)\right\} E_3\left(\int_t^\tau \left\{1 - \exp \left[- \frac{h\nu}{kT(\xi)}\right]\right\} d\xi\right) dt \ . \tag{184}$$

When $B_\nu(t)$ is replaced by

$$\frac{2h\nu^3}{c^2}\left\{\exp\left[\frac{h\nu}{kT(t)}\right]-1\right\}^{-1},$$

and the resulting expression is written in terms of the variable $y = h\nu/kT_e$ and the function $z(t) = [f/B(t)]^{\frac{1}{4}}$, it becomes

$$^{II}\Pi_{1,\nu}(\tau) = \frac{30}{\pi^3 c} F \frac{h}{kT_e} y^3 \left(\int_\tau^\infty e^{-yz(t)} E_3 \left\{\int_\tau^t \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt\right)$$

$$+ \frac{30}{\pi^3 c} F \frac{h}{kT_e} y^3 \left(\int_0^\tau e^{-yz(t)} E_3 \left\{\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt\right). \quad (185)$$

Therefore

$$^{II}\Pi_1(\tau) = \frac{30}{\pi^3 c} F \int_0^\infty \int_\tau^\infty y^3 e^{-yz(t)} E_3 \left\{\int_\tau^t \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt\, dy$$

$$+ \frac{30}{\pi^3 c} F \int_0^\infty \int_0^\tau y^3 e^{-yz(t)} E_3 \left\{\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt\, dy. \quad (186)$$

The second monochromatic principal value, as expressed in Equation 181, is the difference of two terms. The first term is $2\pi/c$ times the mean intensity $J_\nu(\tau)$, of Equation 72. The second term is 1/2 of the first monochromatic principal value. The difference of these two terms is then

$$^{II}\Pi_{2,\nu}(\tau) = \frac{15}{\pi^3 c} F \frac{h}{kT_e} y^3 \left(\int_0^\infty e^{-yz(t)} E_1 \left\{\left|\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right|\right\} dt\right)$$

$$- \frac{15}{\pi^3 c} F \frac{h}{kT_e} y^3 \left(\int_\tau^\infty e^{-yz(t)} E_3 \left\{\int_\tau^t \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt\right)$$

$$- \frac{15}{\pi^3 c} F \frac{h}{kT_e} y^3 \left(\int_0^\tau e^{-yz(t)} E_3 \left\{\int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi\right\} dt\right). \quad (187)$$

Therefore

$$^{\mathrm{II}}\Pi_2(\tau) = {}^{\mathrm{II}}\Pi_3(\tau) = \frac{15}{\pi^3 c} F \int_0^\infty \int_0^\infty y^3 e^{-yz(t)} E_1 \left\{ \left| \int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi \right| \right\} dt\, dy$$

$$- \frac{15}{\pi^3 c} F \int_0^\infty \int_\tau^\infty y^3 e^{-yz(t)} E_3 \left\{ \int_\tau^t \left[1 - e^{-yz(\xi)}\right] d\xi \right\} dt\, dy$$

$$- \frac{15}{\pi^3 c} F \int_0^\infty \int_0^\tau y^3 e^{-yz(t)} E_3 \left\{ \int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi \right\} dt\, dy \quad . \tag{188}$$

The mean radiative stress at the level τ is $4\pi/3c$ times the mean intensity $J(\tau)$, i.e.,

$$^{\mathrm{II}}\overline{P(\tau)} = \frac{30}{\pi^3 c} F \int_0^\infty \int_0^\infty y^3 e^{-yz(t)} E_1 \left\{ \left| \int_t^\tau \left[1 - e^{-yz(\xi)}\right] d\xi \right| \right\} dt\, dy \, . \tag{189}$$

The difference between the normal component of the stress tensor and either one of the tangential components can be denoted by $^{\mathrm{II}}\nabla(\tau)$. It is numerically equal to $^{\mathrm{II}}\Pi_1(\tau) - {}^{\mathrm{II}}\Pi_2(\tau)$. Values of the ratio $^{\mathrm{II}}\nabla(\tau)/{}^{\mathrm{II}}\overline{P(\tau)}$ are listed in Table 10. For the sake of comparison, values of the ratio

$$\frac{^{\mathrm{I}}\nabla\left(^{\mathrm{I}}\tau\right)}{\overline{P\left(^{\mathrm{I}}\tau\right)}} = \frac{\Pi_1\left(^{\mathrm{I}}\tau\right) - \Pi_2\left(^{\mathrm{I}}\tau\right)}{\overline{P\left(^{\mathrm{I}}\tau\right)}}$$

characteristic of the classical gray atmosphere, are also listed in Table 10.

At all levels within the classical gray atmosphere, the normal component of the radiative stress is greater than the tangential components, and $^{\mathrm{I}}\nabla\left(^{\mathrm{I}}\tau\right)/\overline{P\left(^{\mathrm{I}}\tau\right)}$ is a monotonically decreasing function of $^{\mathrm{I}}\tau$ which approaches 0 at large values of $^{\mathrm{I}}\tau$. Within the gray atmosphere of the second kind, the normal component of the radiative stress is greater than the tangential component from the surface down to $\tau = 1.8$. Below this level, each one of the tangential components of the stress is larger than the normal component. From the surface to $\tau = 2.8$, $^{\mathrm{II}}\nabla(\tau)/{}^{\mathrm{II}}\overline{P(\tau)}$ is a decreasing function of τ. It then begins to increase, approaching 0 at large values of τ.

Table 10

The Difference Between the Normal and Tangential Components of the Radiative Stress Tensor Divided by the Mean Radiative Stress.

τ	$\frac{{}^{II}\nabla(\tau)}{{}^{II}P(\tau)}$	${}^{I}\tau$	$\frac{{}^{I}\nabla\ {}^{I}\tau}{P\ {}^{I}\tau}$
0.00	0.337164	0.00	0.345794
0.01	0.300464	0.01	0.307012
0.03	0.253873	0.03	0.258660
0.05	0.222242	0.05	0.226000
0.07	0.197516	0.07	0.200493
0.09	0.177093	0.09	0.179486
0.10	0.168106	0.10	0.170270
0.30	0.072078	0.30	0.073287
0.50	0.036717	0.50	0.038327
0.70	0.019978	0.70	0.021949
0.90	0.011124	0.90	0.013301
1.00	0.008285	1.00	0.010515
1.10	0.006129	1.10	0.008385
1.30	0.003200	1.30	0.005445
1.50	0.001438	1.50	0.003618
1.70	0.000366	1.70	0.002448
1.90	-0.000283	1.90	0.001682
2.00	-0.000503	2.00	0.001401
2.10	-0.000670	2.10	0.001171
2.30	-0.000893	2.30	0.000824
2.50	-0.001009	2.50	0.000586
2.70	-0.001059	2.70	0.000420
2.90	-0.001066	2.90	0.000303
3.00	-0.001059	3.00	0.000258
3.10	-0.001046	3.10	0.000220
3.30	-0.001011	3.30	0.000161
3.50	-0.000966	3.50	0.000118
3.70	-0.000917	3.70	0.000087
3.90	-0.000866	3.90	0.000065
4.00	-0.000840	4.00	0.000056
5.00	-0.000611	5.00	0.000013
10.00	-0.000167	10.00	0.000000
50.00	-0.000006	50.00	0.000000
100.00	-0.000001	100.00	0.000000

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Appendix A

Additional Numerical Results

The following table contains the numerical values of the source function $B(\tau)/F$ and the remainder function $r(\tau)$ at the 602 τ-points between $\tau = 0$ and $\tau = 100$ used in the computation.

	τ	$B(\tau)/F$	$r(\tau)$		τ	$B(\tau)/F$	$r(\tau)$
1	0.0	0.443597	0.624179	61	0.000389	0.444228	0.624677
2	0.000100	0.443760	0.624308	62	0.000398	0.444243	0.624689
3	0.000102	0.443764	0.624311	63	0.000407	0.444258	0.624701
4	0.000105	0.443768	0.624314	64	0.000417	0.444273	0.624713
5	0.000107	0.443772	0.624317	65	0.000427	0.444289	0.624725
6	0.000110	0.443776	0.624320	66	0.000437	0.444305	0.624738
7	0.000112	0.443780	0.624323	67	0.000447	0.444321	0.624751
8	0.000115	0.443784	0.624327	68	0.000457	0.444338	0.624764
9	0.000117	0.443788	0.624330	69	0.000468	0.444355	0.624777
10	0.000120	0.443793	0.624334	70	0.000479	0.444373	0.624791
11	0.000123	0.443797	0.624337	71	0.000490	0.444391	0.624805
12	0.000126	0.443802	0.624341	72	0.000501	0.444409	0.624820
13	0.000129	0.443807	0.624345	73	0.000513	0.444428	0.624834
14	0.000132	0.443812	0.624349	74	0.000525	0.444447	0.624850
15	0.000135	0.443817	0.624353	75	0.000537	0.444467	0.624865
16	0.000138	0.443822	0.624357	76	0.000550	0.444487	0.624881
17	0.000141	0.443827	0.624361	77	0.000562	0.444507	0.624897
18	0.000145	0.443832	0.624365	78	0.000575	0.444528	0.624913
19	0.000148	0.443838	0.624369	79	0.000589	0.444550	0.624930
20	0.000151	0.443843	0.624374	80	0.000603	0.444572	0.624948
21	0.000155	0.443849	0.624378	81	0.000617	0.444594	0.624965
22	0.000158	0.443855	0.624383	82	0.000631	0.444618	0.624983
23	0.000162	0.443861	0.624388	83	0.000646	0.444641	0.625002
24	0.000166	0.443867	0.624392	84	0.000661	0.444665	0.625021
25	0.000170	0.443873	0.624397	85	0.000676	0.444690	0.625040
26	0.000174	0.443880	0.624403	86	0.000692	0.444715	0.625060
27	0.000178	0.443886	0.624408	87	0.000708	0.444741	0.625080
28	0.000182	0.443893	0.624413	88	0.000724	0.444767	0.625101
29	0.000186	0.443900	0.624418	89	0.000741	0.444794	0.625122
30	0.000191	0.443907	0.624424	90	0.000759	0.444822	0.625144
31	0.000195	0.443914	0.624430	91	0.000776	0.444850	0.625166
32	0.000200	0.443922	0.624436	92	0.000794	0.444879	0.625188
33	0.000204	0.443929	0.624441	93	0.000813	0.444909	0.625211
34	0.000209	0.443937	0.624448	94	0.000832	0.444939	0.625235
35	0.000214	0.443945	0.624454	95	0.000851	0.444970	0.625259
36	0.000219	0.443953	0.624460	96	0.000871	0.445002	0.625284
37	0.000224	0.443961	0.624467	97	0.000891	0.445034	0.625309
38	0.000229	0.443969	0.624473	98	0.000912	0.445067	0.625335
39	0.000234	0.443978	0.624480	99	0.000933	0.445101	0.625361
40	0.000240	0.443987	0.624487	100	0.000955	0.445136	0.625388
41	0.000245	0.443996	0.624494	101	0.000977	0.445171	0.625416
42	0.000251	0.444005	0.624502	102	0.001000	0.445207	0.625444
43	0.000257	0.444015	0.624509	103	0.001023	0.445244	0.625473
44	0.000263	0.444024	0.624517	104	0.001047	0.445282	0.625503
45	0.000269	0.444034	0.624525	105	0.001072	0.445321	0.625533
46	0.000275	0.444044	0.624533	106	0.001096	0.445361	0.625564
47	0.000282	0.444055	0.624541	107	0.001122	0.445401	0.625595
48	0.000288	0.444066	0.624549	108	0.001148	0.445443	0.625627
49	0.000295	0.444076	0.624558	109	0.001175	0.445485	0.625660
50	0.000302	0.444087	0.624567	110	0.001202	0.445528	0.625694
51	0.000309	0.444099	0.624576	111	0.001230	0.445573	0.625728
52	0.000316	0.444111	0.624585	112	0.001259	0.445618	0.625763
53	0.000324	0.444122	0.624594	113	0.001288	0.445665	0.625799
54	0.000331	0.444135	0.624604	114	0.001318	0.445712	0.625836
55	0.000339	0.444147	0.624614	115	0.001349	0.445760	0.625874
56	0.000347	0.444160	0.624624	116	0.001380	0.445810	0.625912
57	0.000355	0.444173	0.624634	117	0.001413	0.445861	0.625951
58	0.000363	0.444186	0.624644	118	0.001445	0.445913	0.625991
59	0.000372	0.444200	0.624655	119	0.001479	0.445966	0.626032
60	0.000380	0.444214	0.624666	120	0.001514	0.446020	0.626074

	τ	$B(\tau)/F$	$r(\tau)$		τ	$B(\tau)/F$	$r(\tau)$
121	0.001549	0.446075	0.626117	181	0.006166	0.452988	0.631227
122	0.001585	0.446132	0.626160	182	0.006310	0.453193	0.631372
123	0.001622	0.446190	0.626205	183	0.006457	0.453403	0.631519
124	0.001660	0.446249	0.626250	184	0.006607	0.453616	0.631669
125	0.001698	0.446310	0.626297	185	0.006761	0.453834	0.631822
126	0.001738	0.446372	0.626345	186	0.006918	0.454057	0.631978
127	0.001778	0.446435	0.626393	187	0.007079	0.454284	0.632136
128	0.001820	0.446500	0.626443	188	0.007244	0.454515	0.632297
129	0.001862	0.446566	0.626494	189	0.007413	0.454752	0.632461
130	0.001905	0.446633	0.626545	190	0.007586	0.454993	0.632628
131	0.001950	0.446703	0.626598	191	0.007762	0.455239	0.632797
132	0.001995	0.446773	0.626652	192	0.007943	0.455490	0.632970
133	0.002042	0.446846	0.626708	193	0.008128	0.455746	0.633145
134	0.002089	0.446919	0.626764	194	0.008318	0.456007	0.633323
135	0.002138	0.446995	0.626822	195	0.008511	0.456274	0.633504
136	0.002188	0.447072	0.626880	196	0.008710	0.456545	0.633688
137	0.002239	0.447151	0.626941	197	0.008913	0.456823	0.633875
138	0.002291	0.447232	0.627002	198	0.009120	0.457105	0.634066
139	0.002344	0.447314	0.627064	199	0.009333	0.457394	0.634259
140	0.002399	0.447398	0.627128	200	0.009550	0.457688	0.634455
141	0.002455	0.447485	0.627194	201	0.009772	0.457987	0.634655
142	0.002512	0.447573	0.627260	202	0.010000	0.458293	0.634857
143	0.002570	0.447662	0.627328	203	0.010233	0.458605	0.635063
144	0.002630	0.447754	0.627398	204	0.010471	0.458923	0.635272
145	0.002692	0.447848	0.627469	205	0.010715	0.459247	0.635484
146	0.002754	0.447944	0.627541	206	0.010965	0.459577	0.635699
147	0.002818	0.448042	0.627615	207	0.011220	0.459914	0.635918
148	0.002884	0.448143	0.627691	208	0.011482	0.460258	0.636140
149	0.002951	0.448245	0.627768	209	0.011749	0.460608	0.636365
150	0.003020	0.448350	0.627846	210	0.012023	0.460965	0.636594
151	0.003090	0.448457	0.627926	211	0.012303	0.461328	0.636825
152	0.003162	0.448566	0.628008	212	0.012589	0.461699	0.637061
153	0.003236	0.448678	0.628092	213	0.012882	0.462077	0.637299
154	0.003311	0.448792	0.628177	214	0.013183	0.462463	0.637541
155	0.003388	0.448908	0.628264	215	0.013490	0.462855	0.637787
156	0.003467	0.449028	0.628352	216	0.013804	0.463255	0.638036
157	0.003548	0.449149	0.628443	217	0.014125	0.463663	0.638288
158	0.003631	0.449273	0.628535	218	0.014454	0.464079	0.638544
159	0.003715	0.449400	0.628629	219	0.014791	0.464502	0.638803
160	0.003802	0.449530	0.628725	220	0.015136	0.464934	0.639066
161	0.003890	0.449663	0.628823	221	0.015488	0.465374	0.639332
162	0.003981	0.449798	0.628922	222	0.015849	0.465822	0.639602
163	0.004074	0.449936	0.629024	223	0.016218	0.466279	0.639875
164	0.004169	0.450077	0.629128	224	0.016596	0.466744	0.640152
165	0.004266	0.450221	0.629234	225	0.016982	0.467218	0.640433
166	0.004365	0.450369	0.629341	226	0.017378	0.467701	0.640717
167	0.004467	0.450519	0.629451	227	0.017783	0.468193	0.641005
168	0.004571	0.450672	0.629563	228	0.018197	0.468695	0.641296
169	0.004677	0.450829	0.629677	229	0.018621	0.469206	0.641591
170	0.004786	0.450989	0.629794	230	0.019055	0.469726	0.641890
171	0.004898	0.451153	0.629912	231	0.019498	0.470256	0.642192
172	0.005012	0.451320	0.630033	232	0.019953	0.470797	0.642498
173	0.005129	0.451490	0.630156	233	0.020417	0.471347	0.642808
174	0.005248	0.451664	0.630282	234	0.020893	0.471908	0.643121
175	0.005370	0.451842	0.630409	235	0.021380	0.472479	0.643439
176	0.005495	0.452023	0.630539	236	0.021878	0.473061	0.643760
177	0.005623	0.452208	0.630672	237	0.022387	0.473654	0.644084
178	0.005754	0.452397	0.630807	238	0.022909	0.474258	0.644413
179	0.005888	0.452590	0.630944	239	0.023442	0.474874	0.644745
180	0.006026	0.452787	0.631084	240	0.023988	0.475501	0.645082

	τ	B(τ)/F	r(τ)		τ	B(τ)/F	r(τ)
241	0.024547	0.476140	0.645422	301	0.097724	0.548815	0.674506
242	0.025119	0.476791	0.645766	302	0.100000	0.550898	0.675160
243	0.025704	0.477454	0.646114	303	0.102329	0.553023	0.675821
244	0.026303	0.478129	0.646466	304	0.104713	0.555190	0.676486
245	0.026915	0.478818	0.646821	305	0.107152	0.557401	0.677158
246	0.027542	0.479519	0.647181	306	0.109648	0.559655	0.677834
247	0.028184	0.480233	0.647545	307	0.112202	0.561955	0.678516
248	0.028840	0.480962	0.647913	308	0.114815	0.564301	0.679204
249	0.029512	0.481703	0.648285	309	0.117490	0.566693	0.679896
250	0.030200	0.482459	0.648661	310	0.120226	0.569134	0.680593
251	0.030903	0.483230	0.649042	311	0.123027	0.571623	0.681295
252	0.031623	0.484015	0.649427	312	0.125893	0.574162	0.682002
253	0.032359	0.484814	0.649815	313	0.128825	0.576752	0.682713
254	0.033113	0.485630	0.650209	314	0.131826	0.579393	0.683429
255	0.033884	0.486460	0.650606	315	0.134896	0.582087	0.684149
256	0.034674	0.487307	0.651008	316	0.138038	0.584835	0.684874
257	0.035481	0.488170	0.651415	317	0.141254	0.587638	0.685603
258	0.036308	0.489049	0.651826	318	0.144544	0.590497	0.686335
259	0.037154	0.489945	0.652241	319	0.147911	0.593413	0.687072
260	0.038019	0.490859	0.652661	320	0.151356	0.596388	0.687812
261	0.038905	0.491790	0.653086	321	0.154882	0.599422	0.688555
262	0.039811	0.492740	0.653515	322	0.158489	0.602517	0.689302
263	0.040738	0.493707	0.653950	323	0.162181	0.605673	0.690052
264	0.041687	0.494694	0.654389	324	0.165959	0.608893	0.690806
265	0.042658	0.495699	0.654833	325	0.169824	0.612178	0.691562
266	0.043652	0.496724	0.655281	326	0.173780	0.615529	0.692321
267	0.044668	0.497770	0.655735	327	0.177828	0.618947	0.693082
268	0.045709	0.498835	0.656194	328	0.181970	0.622433	0.693846
269	0.046774	0.499922	0.656658	329	0.186209	0.625990	0.694612
270	0.047863	0.501030	0.657128	330	0.190546	0.629618	0.695380
271	0.048978	0.502159	0.657602	331	0.194984	0.633320	0.696150
272	0.050119	0.503311	0.658082	332	0.199526	0.637096	0.696921
273	0.051286	0.504485	0.658567	333	0.204174	0.640948	0.697694
274	0.052481	0.505683	0.659058	334	0.208930	0.644878	0.698469
275	0.053703	0.506905	0.659554	335	0.213796	0.648888	0.699244
276	0.054954	0.508150	0.660056	336	0.218776	0.652979	0.700021
277	0.056234	0.509420	0.660563	337	0.223872	0.657153	0.700798
278	0.057544	0.510716	0.661076	338	0.229087	0.661412	0.701575
279	0.058884	0.512037	0.661595	339	0.234423	0.665757	0.702353
280	0.060256	0.513385	0.662120	340	0.239883	0.670191	0.703132
281	0.061660	0.514759	0.662650	341	0.245471	0.674715	0.703910
282	0.063096	0.516161	0.663186	342	0.251189	0.679331	0.704687
283	0.064565	0.517590	0.663728	343	0.257040	0.684042	0.705465
284	0.066069	0.519049	0.664276	344	0.263027	0.688849	0.706242
285	0.067608	0.520536	0.664830	345	0.269153	0.693754	0.707017
286	0.069183	0.522053	0.665390	346	0.275423	0.698760	0.707792
287	0.070795	0.523600	0.665955	347	0.281838	0.703869	0.708565
288	0.072444	0.525179	0.666527	348	0.288403	0.709083	0.709337
289	0.074131	0.526789	0.667105	349	0.295121	0.714405	0.710107
290	0.075858	0.528431	0.667689	350	0.301995	0.719836	0.710875
291	0.077625	0.530106	0.668279	351	0.309030	0.725379	0.711641
292	0.079433	0.531814	0.668875	352	0.316228	0.731037	0.712404
293	0.081283	0.533557	0.669477	353	0.323594	0.736813	0.713164
294	0.083176	0.535335	0.670084	354	0.331131	0.742708	0.713921
295	0.085114	0.537148	0.670698	355	0.338844	0.748725	0.714676
296	0.087096	0.538997	0.671318	356	0.346737	0.754868	0.715426
297	0.089125	0.540884	0.671944	357	0.354813	0.761139	0.716173
298	0.091201	0.542808	0.672576	358	0.363078	0.767540	0.716916
299	0.093325	0.544771	0.673213	359	0.371535	0.774076	0.717654
300	0.095499	0.546773	0.673857	360	0.380189	0.780748	0.718388

	τ	B(τ)/F	r(τ)		τ	B(τ)/F	r(τ)
361	0.389045	0.787560	0.719118	421	1.548817	1.629167	0.743559
362	0.398107	0.794514	0.719841	422	1.584893	1.654781	0.743523
363	0.407380	0.801615	0.720560	423	1.621810	1.680981	0.743472
364	0.416869	0.808866	0.721273	424	1.659587	1.707782	0.743406
365	0.426580	0.816269	0.721980	425	1.698244	1.735197	0.743325
366	0.436516	0.823828	0.722680	426	1.737801	1.763242	0.743229
367	0.446684	0.831548	0.723374	427	1.778279	1.791932	0.743120
368	0.457088	0.839431	0.724061	428	1.819701	1.821282	0.742996
369	0.467735	0.847480	0.724741	429	1.862087	1.851307	0.742858
370	0.478630	0.855701	0.725414	430	1.905461	1.882025	0.742707
371	0.489779	0.864097	0.726078	431	1.949845	1.913451	0.742543
372	0.501187	0.872671	0.726735	432	1.995262	1.945603	0.742366
373	0.512861	0.881428	0.727383	433	2.041738	1.978498	0.742176
374	0.524807	0.890373	0.728022	434	2.089296	2.012154	0.741974
375	0.537032	0.899508	0.728652	435	2.137962	2.046588	0.741760
376	0.549541	0.908840	0.729273	436	2.187762	2.081820	0.741535
377	0.562341	0.918372	0.729885	437	2.238721	2.117869	0.741299
378	0.575440	0.928108	0.730486	438	2.290868	2.154753	0.741052
379	0.588844	0.938054	0.731077	439	2.344229	2.192493	0.740794
380	0.602560	0.948214	0.731658	440	2.398833	2.231110	0.740527
381	0.616595	0.958594	0.732228	441	2.454709	2.270624	0.740251
382	0.630957	0.969198	0.732787	442	2.511886	2.311056	0.739965
383	0.645654	0.980032	0.733334	443	2.570396	2.352429	0.739670
384	0.660693	0.991101	0.733869	444	2.630268	2.394765	0.739368
385	0.676083	1.002410	0.734393	445	2.691535	2.438086	0.739058
386	0.691831	1.013966	0.734904	446	2.754229	2.482416	0.738740
387	0.707946	1.025773	0.735403	447	2.818383	2.527779	0.738416
388	0.724436	1.037837	0.735889	448	2.884032	2.574200	0.738085
389	0.741310	1.050166	0.736362	449	2.951209	2.621703	0.737748
390	0.758578	1.062764	0.736821	450	3.019952	2.670314	0.737406
391	0.776247	1.075639	0.737267	451	3.090295	2.720060	0.737059
392	0.794328	1.088796	0.737700	452	3.162278	2.770967	0.736707
393	0.812831	1.102243	0.738118	453	3.235937	2.823063	0.736351
394	0.831764	1.115985	0.738522	454	3.311311	2.876375	0.735992
395	0.851138	1.130031	0.738911	455	3.388442	2.930933	0.735629
396	0.870964	1.144388	0.739286	456	3.467369	2.986766	0.735263
397	0.891251	1.159062	0.739647	457	3.548134	3.043904	0.734895
398	0.912011	1.174061	0.739992	458	3.630781	3.102377	0.734526
399	0.933254	1.189393	0.740322	459	3.715352	3.162217	0.734154
400	0.954993	1.205066	0.740636	460	3.801894	3.223457	0.733782
401	0.977237	1.221087	0.740936	461	3.890451	3.286129	0.733409
402	1.000000	1.237466	0.741219	462	3.981072	3.350266	0.733035
403	1.023293	1.254210	0.741487	463	4.073803	3.415904	0.732662
404	1.047129	1.271329	0.741739	464	4.168694	3.483077	0.732289
405	1.071519	1.288831	0.741975	465	4.265795	3.551821	0.731917
406	1.096478	1.306726	0.742195	466	4.365158	3.622174	0.731545
407	1.122018	1.325022	0.742399	467	4.466836	3.694172	0.731176
408	1.148154	1.343729	0.742587	468	4.570882	3.767855	0.730808
409	1.174898	1.362858	0.742758	469	4.677351	3.843261	0.730442
410	1.202264	1.382418	0.742914	470	4.786301	3.920432	0.730078
411	1.230269	1.402419	0.743053	471	4.897788	3.999409	0.729717
412	1.258925	1.422872	0.743176	472	5.011872	4.080232	0.729359
413	1.288250	1.443788	0.743282	473	5.128614	4.162947	0.729004
414	1.318257	1.465178	0.743373	474	5.248075	4.247596	0.728652
415	1.348963	1.487054	0.743447	475	5.370318	4.334226	0.728303
416	1.380384	1.509426	0.743506	476	5.495409	4.422881	0.727959
417	1.412538	1.532307	0.743548	477	5.623413	4.513611	0.727618
418	1.445440	1.555709	0.743574	478	5.754399	4.606462	0.727281
419	1.479108	1.579645	0.743585	479	5.888437	4.701484	0.726948
420	1.513561	1.604126	0.743580	480	6.025596	4.798728	0.726620

	τ	B(τ)/F	r(τ)		τ	B(τ)/F	r(τ)
481	6.165950	4.898246	0.726296	541	24.547089	17.953264	0.714667
482	6.309573	5.000091	0.725976	542	25.118864	18.359549	0.714570
483	6.456542	5.104317	0.725662	543	25.703958	18.775301	0.714474
484	6.606934	5.210978	0.725352	544	26.302680	19.200740	0.714380
485	6.760830	5.320133	0.725046	545	26.915348	19.636091	0.714289
486	6.918310	5.431839	0.724746	546	27.542287	20.081586	0.714199
487	7.079458	5.546155	0.724450	547	28.183829	20.537461	0.714110
488	7.244360	5.663142	0.724159	548	28.840315	21.003957	0.714024
489	7.413102	5.782863	0.723873	549	29.512092	21.481321	0.713939
490	7.585776	5.905380	0.723592	550	30.199517	21.969808	0.713856
491	7.762471	6.030759	0.723316	551	30.902954	22.469675	0.713774
492	7.943282	6.159067	0.723045	552	31.622777	22.981188	0.713694
493	8.128305	6.290372	0.722778	553	32.359366	23.504619	0.713615
494	8.317638	6.424742	0.722517	554	33.113112	24.040243	0.713538
495	8.511380	6.562251	0.722260	555	33.884416	24.588347	0.713463
496	8.709636	6.702970	0.722008	556	34.673685	25.149220	0.713388
497	8.912509	6.846974	0.721761	557	35.481339	25.723159	0.713315
498	9.120108	6.994341	0.721519	558	36.307805	26.310469	0.713244
499	9.332543	7.145147	0.721281	559	37.153523	26.911461	0.713173
500	9.549926	7.299473	0.721048	560	38.018940	27.526455	0.713104
501	9.772372	7.457401	0.720820	561	38.904514	28.155775	0.713037
502	10.000000	7.619014	0.720596	562	39.810717	28.799757	0.712970
503	10.232930	7.784399	0.720376	563	40.738028	29.458740	0.712905
504	10.471285	7.953643	0.720161	564	41.686938	30.133075	0.712840
505	10.715193	8.126836	0.719950	565	42.657952	30.823119	0.712777
506	10.964782	8.304069	0.719744	566	43.651583	31.529239	0.712715
507	11.220185	8.485437	0.719541	567	44.668359	32.251807	0.712654
508	11.481536	8.671036	0.719343	568	45.708819	32.991208	0.712594
509	11.748976	8.860964	0.719149	569	46.773514	33.747834	0.712535
510	12.022644	9.055323	0.718959	570	47.863009	34.522086	0.712477
511	12.302688	9.254214	0.718772	571	48.977882	35.314373	0.712420
512	12.589254	9.457744	0.718590	572	50.118723	36.125118	0.712364
513	12.882496	9.666021	0.718411	573	51.286138	36.954748	0.712309
514	13.182567	9.879154	0.718236	574	52.480746	37.803705	0.712255
515	13.489629	10.097258	0.718065	575	53.703180	38.672437	0.712202
516	13.803843	10.320447	0.717897	576	54.954087	39.561407	0.712150
517	14.125375	10.548840	0.717733	577	56.234133	40.471085	0.712098
518	14.454398	10.782559	0.717572	578	57.543994	41.401954	0.712048
519	14.791084	11.021726	0.717414	579	58.884366	42.354507	0.711998
520	15.135612	11.266469	0.717260	580	60.255959	43.329249	0.711949
521	15.488166	11.516918	0.717109	581	61.659500	44.326697	0.711901
522	15.848932	11.773205	0.716961	582	63.095734	45.347380	0.711854
523	16.218101	12.035467	0.716816	583	64.565423	46.391839	0.711807
524	16.595869	12.303842	0.716674	584	66.069345	47.460628	0.711762
525	16.982437	12.578473	0.716535	585	67.608298	48.554313	0.711717
526	17.378008	12.859505	0.716399	586	69.183097	49.673476	0.711673
527	17.782794	13.147087	0.716266	587	70.794578	50.818708	0.711630
528	18.197009	13.441372	0.716136	588	72.443596	51.990617	0.711587
529	18.620871	13.742516	0.716008	589	74.131024	53.189824	0.711545
530	19.054607	14.050679	0.715883	590	75.857758	54.416967	0.711504
531	19.498446	14.366024	0.715761	591	77.624712	55.672694	0.711464
532	19.952623	14.688717	0.715641	592	79.432823	56.957671	0.711425
533	20.417379	15.018931	0.715524	593	81.283052	58.272581	0.711386
534	20.892961	15.356841	0.715409	594	83.176377	59.618121	0.711348
535	21.379621	15.702624	0.715296	595	85.113804	60.995003	0.711310
536	21.877616	16.056466	0.715186	596	87.096359	62.403958	0.711274
537	22.387211	16.418553	0.715078	597	89.125094	63.845733	0.711238
538	22.908677	16.789078	0.714972	598	91.201084	65.321092	0.711202
539	23.442288	17.168236	0.714868	599	93.325430	66.830818	0.711168
540	23.988329	17.556230	0.714767	600	95.499259	68.375711	0.711134
				601	97.723722	69.956590	0.711101
				602	100.000000	71.574293	0.711068

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